2. IN ONE SIDE, OUT THE OTHER
ANGLES AND TRIANGLES

These are the first steps. They are tentative. But it is right to be cautious. It is so difficult keeping intuition from making unjustified leaps. The two main theorems in this lesson, Pasch's Lemma and the Crossbar Theorem, are good examples of this. Neither can be found in Euclid's Elements. They just seem so obvious that I guess it didn't occur to him that they needed to be proved (his framework of postulates would not allow him to prove those results anyway). The kind of intersections that they guarantee are essential to many future results, though, so we must not overlook them.

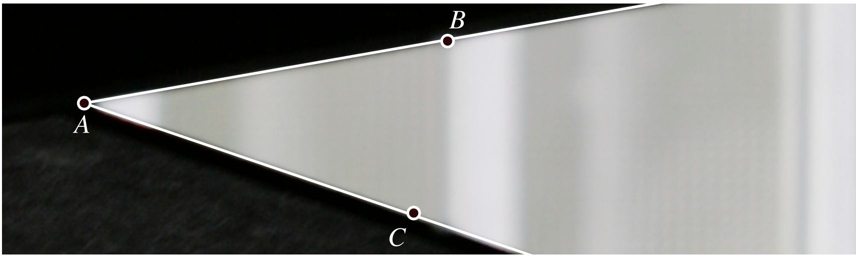
Angles and Triangles

In the last lesson we defined ray and segment. They are the most elementary of objects, defined directly from the undefined terms. Now in this lesson, another layer: angles and triangles, which are built from rays and segments.

DEF: ANGLE
An *angle* consists of a (unordered) pair of non-opposite rays with the same endpoint. The mutual endpoint is called the *vertex* of the angle.

Let's talk notation. If the two rays are $AB \rightarrow$ and $AC \rightarrow$, then the angle they form is written $\angle BAC$, with the endpoint listed in the middle spot. There's more than one way to indicate that angle though. For one, it does not matter which order the rays are taken, so $\angle CAB$ points to the same angle as $\angle BAC$. And if B' is on $AB\rightarrow$ and C' is on $AC\rightarrow$ (not the endpoint of course), then $\angle B'AC'$ is the same as $\angle BAC$ too. Frequently, it is clear in the problem that you only care about one angle at a particular vertex. On those occasions you can often get away with the abbreviation $\angle A$ in place of the full $\angle BAC$. Just as a line divides the plane into two sides, so too does an angle. In this case the two parts are the interior and the exterior of the angle.

DEF: ANGLE INTERIOR
A point lies *in the interior* or is an *interior point* of $\angle BAC$ if it is on the same side of $\leftarrow AB \rightarrow$ as C and same side of $\leftarrow AC \rightarrow$ as B. A point which does not lie in the interior of the angle and does not lie on either of the rays composing the angle is *exterior* to the angle and is called an *exterior point*.



$\angle BAC$. *The light region is the interior. The dark the exterior.*

The last definition in this section is that of the triangle. Just as an angle is formed by joining two rays at their mutual endpoint, a triangle is formed by joining three segments at mutual endpoints.

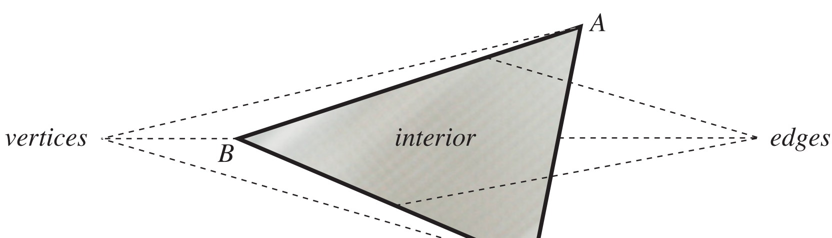
DEF: TRIANGLE

A *triangle* is an (unordered) triple of non-colinear points and the points on the segments between each of the three pairs of points. Each of the three points is called a *vertex* of the triangle. Each of the three segments is called a *side* or *edge* of the triangle.

If A, B, and C are non-colinear points then we write $\triangle ABC$ for the triangle. The ordering of the three vertices does not matter, so there is more than one way to write a given triangle:

$$\triangle ABC = \triangle ACB = \triangle BAC = \triangle BCA = \triangle CAB = \triangle CBA.$$

The three sides of $\triangle ABC$ are AB, AC, and BC. The three angles $\angle ABC$, $\angle BCA$ and $\angle CAB$ are called the *interior angles* of $\triangle ABC$. A point which is in the interior of all the three of the interior angles is said to be *inside* the triangle. Together they form the *interior* of the triangle. Points which are not inside the triangle and aren't on the triangle itself, are said to be *outside* the triangle. They make the *exterior* of the triangle.



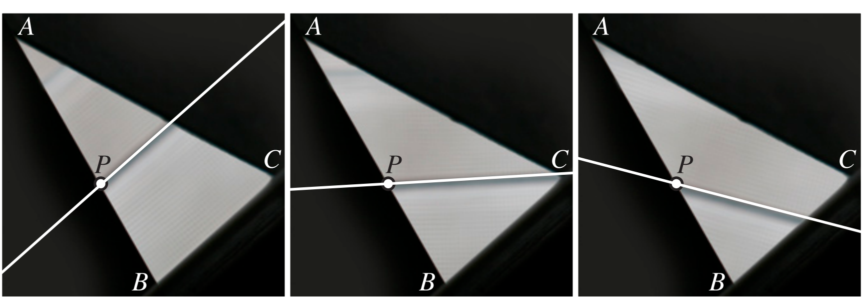
Parts of a triangle

A Line Passes Through It

The rest of this lesson is dedicated to three fundamental theorems. The first, a result about lines crossing triangles is called Pasch's Lemma after Moritz Pasch, a nineteenth century German mathematician whose works are a precursor to Hilbert's. It is a direct consequence of the Plane Separation Axiom. The second result, the Crossbar Theorem, is a bit more difficult. It deals with lines crossing through the vertex of an angle. The third says that rays with a common endpoint can be ordered in a consistent way, in the same way that points on a line can be ordered.

> PASCH'S LEMMA
> If a line intersects a side of a triangle at a point other than a vertex, then it must intersect another side of the triangle. If a line intersects all three sides of a triangle, then it must intersect two of the sides at a vertex.

Proof. Suppose that a line ℓ intersects side AB of $\triangle ABC$ at a point P other than the endpoints. If ℓ also passes through C, then that's the other intersection; in this case ℓ does pass through all three sides of of the triangle, but it passes through two of them at a vertex. Now what if ℓ does not pass through C? There are only two possibilities: either C is on the same side of ℓ as A, or it is on the opposite side of ℓ from A. This is where the Plane Separation Axiom comes to the rescue. Because P is between A and B, those two points have to be on opposite sides of ℓ. Thus, if C is on the same side of ℓ as A, then it is on the opposite side of ℓ from B, and so ℓ intersects BC but not AC. On the other hand, if C is on the opposite side of ℓ from A, then it is on the same side of ℓ as B, so ℓ intersects AC but not BC. Either way, ℓ intersects two of the three sides of the triangle. □



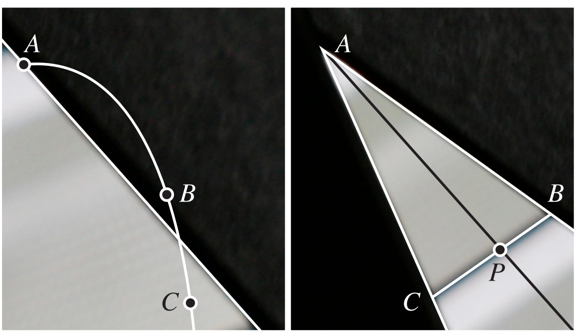
ℓ passes through AC *ℓ passes through C* *ℓ passes through BC*

As I mentioned at the start of the section, the proof of the Crossbar Theorem is more challenging. I think it is helpful to separate out one small part into the following lemma.

LEMMA

If A is a point on line ℓ, and B is a point which is not on ℓ, then all the points of $AB\rightarrow$ (and therefore all the points of AB) except A are on the same side of ℓ as B.

Proof. If C is any point on $AB\rightarrow$ other than A or B, then C has to be on the same side of A as B, and so either $A * B * C$ or $A * C * B$. Either way, $\leftarrow AC \rightarrow$ and ℓ intersect at the point A, but that point of intersection does not lie between B and C. Hence B and C are on the same side of ℓ. □



(l) The lemma says that a ray cannot recross a line like this. (r) The Crossbar Theorem guarantees the existence of the point P.

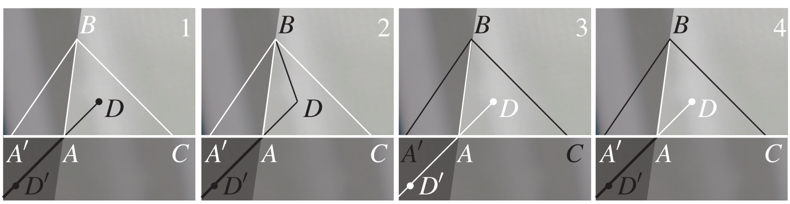
THE CROSSBAR THEOREM

If D is an interior point of angle $\angle BAC$, then the ray $AD\rightarrow$ intersects the segment BC.

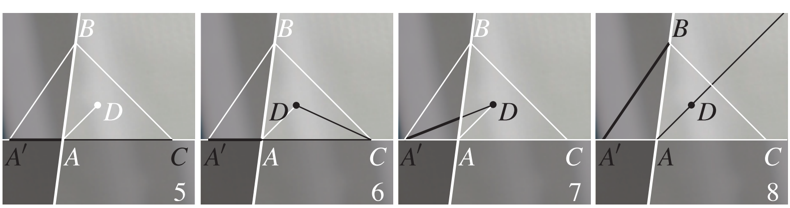
Proof. If you take a couple minutes to try to prove this for yourself, you will probably find yourself thinking– hey, this seems awfully similar to Pasch's Lemma– we could use $\triangle ABC$ for the triangle and $\leftarrow AD \rightarrow$ for the line. The problem is that one pesky condition in Pasch's Lemma: the given intersection of the line and the triangle can't be at a vertex. In the situation we have here, the ray in question $AD\rightarrow$ *does* pass through the vertex. Still, the basic idea is sound. The actual proof does use Pasch's Lemma, we just have to bump the triangle a little bit so that $AD\rightarrow$ doesn't cross through the vertex.

According to the second axiom of order, there are points on the opposite side of A from C. Let A' be one of them. Now $\leftarrow AD \rightarrow$ intersects the side $A'C$ of the triangle $\triangle A'BC$. By Pasch's Lemma, $\leftarrow AD \rightarrow$ must intersect one of the other two sides of triangle, either $A'B$ or BC. There are two scenarios to cause concern. First, what if $\leftarrow AD \rightarrow$ crosses $A'B$ instead of BC? And second, what if $\leftarrow AD \rightarrow$ does cross BC, but the intersection is on $(AD\rightarrow)^{op}$ instead of $AD\rightarrow$ itself?

I think it is easier to rule out the second scenario first so let's start there. (1)If D' is any point on $(AD\rightarrow)^{op}$, then it is on the opposite side of A from D. Therefore D' and D are on opposite sides of $\leftarrow A'C \rightarrow$. (2)Since D is an interior point, it is on the same side of $\leftarrow A'C \rightarrow$ as B, and so D' and B are on opposite sides of $A'C$. (3)By the previous lemma, all the points of $A'B$ and of BC are on the same side of $\leftarrow A'C \rightarrow$ as B. (4)Therefore they are on the opposite side of $\leftarrow A'C \rightarrow$ fom D', so no point of $(AD\rightarrow)^{op}$ may lie on either $A'B$ or BC.

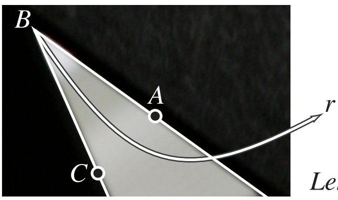


With the opposite ray ruled out entirely, we now just need to make sure that $AD\rightarrow$ does not intersect $A'B$. (5)Points A' and C are on opposite sides of $\leftarrow AB \rightarrow$. (6)Because D is an interior point, D and C are on the same side of $\leftarrow AB \rightarrow$. (7)Therefore A' and D are on opposite sides of $\leftarrow AB \rightarrow$. (8)Using the preceding lemma, all the points of $A'B$ are on opposite sides of $\leftarrow AB \rightarrow$ from all the points of $AD\rightarrow$. This means that $AD\rightarrow$ cannot intersect $A'B$, so it must intersect BC. □



The Crossbar Theorem provides a essential conduit between the notion of *between* for points and *interior* for angles. I would like to use that conduit in the next theorem, which is the angle interior analog to the ordering of points theorem in the last lesson. First let me state a useful lemma.

LEMMA 2
Consider an angle $\angle ABC$ and a ray r whose endpoint is B. Either all the points of r other than B lie in the interior of $\angle ABC$, or none of them do.



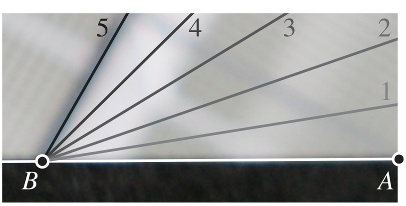
Lemma 2. Rays cannot do this.

I am going to leave the proof of this lemma to you, the reader. It is a relatively straightforward proof, and lemma 1 should provide some useful clues. Now on to the theorem.

THM: ORDERING RAYS
Consider $n \geq 2$ rays with a common basepoint B which are all on the same side of a line $\leftarrow AB \rightarrow$ through B. There is an ordering of them:

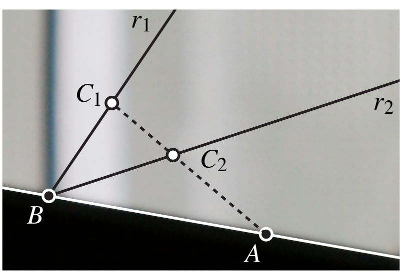
$$r_1,\ r_2,\ \ldots, r_n$$

so that if $i < j$ then r_i is in the interior of the angle formed by $BA\rightarrow$ and r_j.



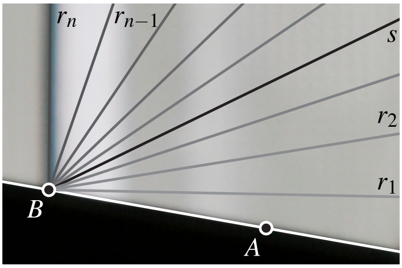
An ordering of five rays and five angles so that each ray is in the interior of all of the subsequent angles.

Proof. I am going to use a proof by induction. First consider the case of just $n = 2$ rays, r_1 and r_2. If r_1 lies in the interior of the angle formed by $BA\rightarrow$ and r_2, then we've got it. Let's suppose, though, that r_1 does not lie in the interior of that angle. There are two requirements for r_1 to lie in the interior: (1) it has to be on the same side of $\leftarrow AB \rightarrow$ as r_2 and (2) it has to be the same side of r_2 as A. From the very statement of the theorem, we can see that r_1 has to satisfy the first requirement, so if r_1 is not in the interior, the problem has got to be with the second requirement. That means that any point C_1 on r_1 has to be on the opposite side of r_2 from A– that is, the line containing r_2 must intersect AC_1. Actually we can be a little more specific about where this intersection occurs: you see, AC_1 and r_2^{op} are on opposite sides of $\leftarrow AB\rightarrow$ so they cannot intersect. Therefore the intersection is not on r_2^{op}– it has to be on r_2 itself. Call this intersection point C_2. Then $A * C_2 * C_1$ so C_2 is on the same side of r_1 as A. Therefore r_2 is on the same side of r_1 as A, and so r_2 is in the interior of the angle formed by $BA\rightarrow$ and r_1. Then it is just a matter of switching the labeling of r_1 and r_2 to get the desired result.



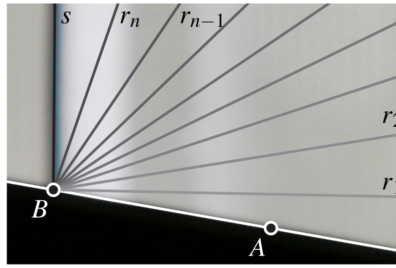
The base case: what happens if r_1 is not in the interior of the angle formed by $BA\rightarrow$ and r_2?

Now let's tackle the inductive step. Assume that any n rays can be put in order and consider a set of $n+1$ rays all sharing a common endpoint B and on the same side of the line $\leftarrow AB\rightarrow$. Take n of those rays and put them in order as $r_1, r_2, \ldots, r_n$. That leaves just one more ray– call it s. What I would like to do is to compare s to what is currently the "outermost" ray, r_n. One of two things can happen: either [1] s lies in the interior of the angle formed by $BA\rightarrow$ and r_n, or [2] it doesn't, and in this case, as we saw in the proof of the base case, that means that r_n lies in the interior of the angle formed by $BA\rightarrow$ and s. Our path splits now, as we consider the two cases.



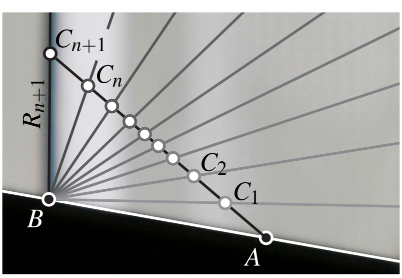
[1] Here r_n is the outermost ray, so let's relabel it as R_{n+1}. The remaining rays r_1, r_2, …, r_{n-1} and s are all in the interior of the angle formed by $BA\rightarrow$ and R_{n+1}. Therefore, if C_{n+1} is any point on R_{n+1} (other than B) then each of r_1, r_2, …, r_{n-1} and s intersect the segment AC_{n+1} (this is the Crossbar Theorem in action). We can put all of those intersection points in order

$$A * C_1 * C_2 * \cdots * C_n * C_{n+1}.$$



[2] In this case, we will eventually see that s is the outermost ray, but all we know at the outset is that it is farther out than r_n. Let's relabel s as R_{n+1} and let C_{n+1} be a point on this ray. Since r_n is in the interior of the angle formed by $BA\rightarrow$ and R_{n+1}, by the Crossbar Theorem, r_n must intersect AC_{n+1}. Let C_n be this intersection point. But we know that $r_1, r_2, \ldots, r_{n-1}$ lie in the interior of the angle formed by $BA\rightarrow$ and R_n, so AC_n must intersect each of $r_1, r_2, \ldots, r_n$. We can put all of those intersection points in order

$$A * C_1 * C_2 * \cdots * C_n * C_{n+1}.$$



Once the outermost ray is identified, a line connecting that ray to A intersects all the other rays (because of the Crossbar Theorem).

With the rays sorted and the intersections marked, the two strands of the proofs merge. Label the ray with point C_i as R_i. Then, for any $i < j$, C_i is on the same side of C_j as A, and so R_i is in the interior of the angle formed by $BA\rightarrow$ and C_j. This is the ordering that we want. $\square$

Exercises

1. Prove that there are points in the interior of any angle. Similarly, prove that there are points in the interior of any triangle.

2. Suppose that a line ℓ intersects a triangle at two points P and Q. Prove that all the points on the segment PQ other than the endpoints P and Q are in the interior of the triangle.

3. We have assumed Plane Separation as an axiom and used it to prove Pasch's Lemma. Try to reverse that– in other words, assume Pasch's Lemma and prove the Plane Separation Axiom.

4. Let P be a point in the interior of $\angle BAC$. Prove that all of the points of $AP\rightarrow$ other than A are also in the interior of $\angle BAC$. Prove that none of the points of $(AP\rightarrow)^{op}$ are in the interior of $\angle BAC$.

5. Prove Lemma 2.

6. A model for a non-neutral geometry: $\mathbb{Q}^2$. We alter the standard Euclidean model $\mathbb{R}^2$ so that the only points are those with rational coordinates. The only lines are those that pass through at least two rational points. Incidence and order are as in the Euclidean model. Demonstrate that this models a geometry which satisfies all the axioms of incidence and order except the Plane Separation Axiom. Show that Pasch's Lemma and the Crossbar Theorem do not hold in this geometry.

References

I got my proof of the Crossbar Theorem from Moise's book on Euclidean geometry [1].

[1] Edwin E. Moise. *Elementary Geometry from an Advanced Standpoint*. Addison Wesley Publishing Company, Reading, Massachusetts, 2nd edition, 1974.