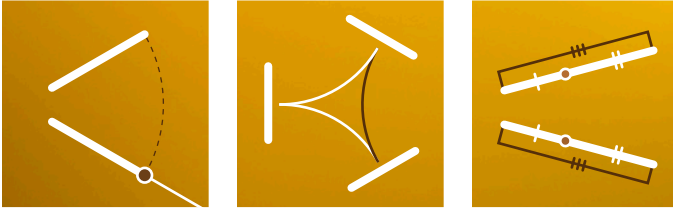




**3. CONGRUENCE VERSE I**  
**OBJECTIVE: SAS AND ASA**



*Cg1 The Segment Construction Axiom* If  $A$  and  $B$  are distinct points and if  $A'$  is any point, then for each ray  $r$  with endpoint  $A'$ , there is a unique point  $B'$  on  $r$  such that  $AB \simeq A'B'$ .

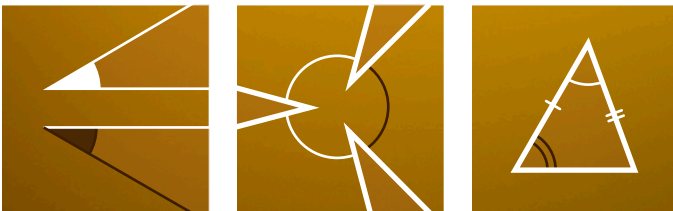
*Cg2* Segment congruence is reflexive (every segment is congruent to itself), symmetric (if  $AA' \simeq BB'$  then  $BB' \simeq AA'$ ), and transitive (if  $AA' \simeq BB'$  and  $BB' \simeq CC'$ , then  $AA' \simeq CC'$ ).

*Cg3 The Segment Addition Axiom* If  $A * B * C$  and  $A' * B' * C'$ , and if  $AB \simeq A'B'$  and  $BC \simeq B'C'$ , then  $AC \simeq A'C'$ .

*Cg4 The Angle Construction Axiom* Given  $\angle BAC$  and any ray  $A'B' \rightarrow$ , there is a unique ray  $A'C' \rightarrow$  on a given side of the line  $\leftarrow A'B' \rightarrow$  such that  $\angle BAC \simeq \angle B'A'C'$ .

*Cg5* Angle congruence is reflexive (every angle is congruent to itself), symmetric (if  $\angle A \simeq \angle B$ , then  $\angle B \simeq \angle A$ ), and transitive (if  $\angle A \simeq \angle B$  and  $\angle B \simeq \angle C$ , then  $\angle A \simeq \angle C$ ).

*Cg6 The Side Angle Side (S·A·S) Axiom.* Consider two triangles:  $\triangle ABC$  and  $\triangle A'B'C'$ . If  $AB \simeq A'B'$ ,  $\angle B \simeq \angle B'$ , and  $BC \simeq B'C'$ , then  $\angle A \simeq \angle A'$ .

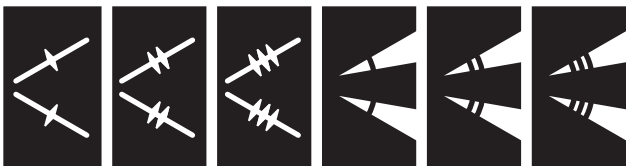


I think this is the lesson where the geometry we are doing starts to look like the geometry you know. I don't think your typical high school geometry class covers Pasch's Lemma or the Crossbar Theorem, but I'm pretty sure that it does cover congruence of triangles. And that is what we are going to do in the next three lessons.

## Axioms of Congruence

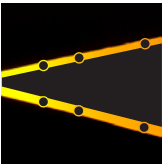
Points, lines, segments, rays, angles, triangles— we are starting to pile up a lot of objects here. At some point you are probably going to want to compare them to each other. You might have two different triangles in different locations, different orientations, but they have essentially the same shape, so you want to say that for practical purposes, they are equivalent. Well, congruence is a way to do that. Congruence, if you recall, is one of the undefined terms in Hilbert's system. Initially it describes a relation between a pair of segments or a pair of angles, so that we can say, for instance, that two segments are or are not congruent, or that two angles are or are not congruent. Later, the term is extended so that we can talk about congruence of triangles and other more general shapes. The notation used to indicate that two things (segments, angles, whatever) are congruent is  $\cong$ . In Hilbert's system, there are six axioms of congruence. Three deal with congruence of segments, two deal with congruence of angles, and one involves both segments and angles.

The first and fourth of these make it possible to construct congruent copies of segments and angles wherever we want. They are a little reminiscent of Euclid's postulates in that way. The second and fifth axioms tell us that congruence is an equivalence relation. The third and sixth— well, I suppose that in a way they form a pair too— both deal with three points and the segments that have them as their endpoints. In the third axiom, the points are colinear, while in the sixth they are not. There is a more direct counterpart to the third axiom though, a statement which does for angles what the Segment Addition Axiom does for segments. It is called the Angle Addition Theorem and we will prove it in lesson 5.



*I use a variety of symbols to mark segment and angle congruence.*

Any time you throw something new into the mix, you probably want to figure out how it intermingles with what has come before. How does the new fit with the old? I realize that is a pretty vague question, but a more precise statement really depends upon the context. In our current situation, we have just added congruence to a system that already had incidence and order. The axioms of congruence themselves provide some basic connections between congruence and incidence and order. I think the most important remaining connection between congruence, incidence, and order is the Triangle Inequality, but that result is still a little ways away. In the meantime, the next theorem provides one more connection.



#### CONGRUENCE AND ORDER

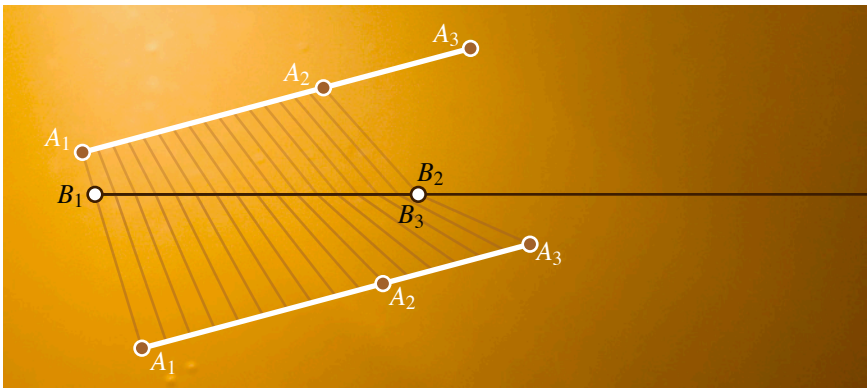
Suppose that  $A_1 * A_2 * A_3$  and that  $B_3$  is a point on the ray  $B_1B_2 \rightarrow$ . If  $A_1A_2 \simeq B_1B_2$  and  $A_1A_3 \simeq B_1B_3$ , then  $B_1 * B_2 * B_3$ .

*Proof.* Since  $B_3$  is on  $B_1B_2 \rightarrow$  one of three things is going to happen:

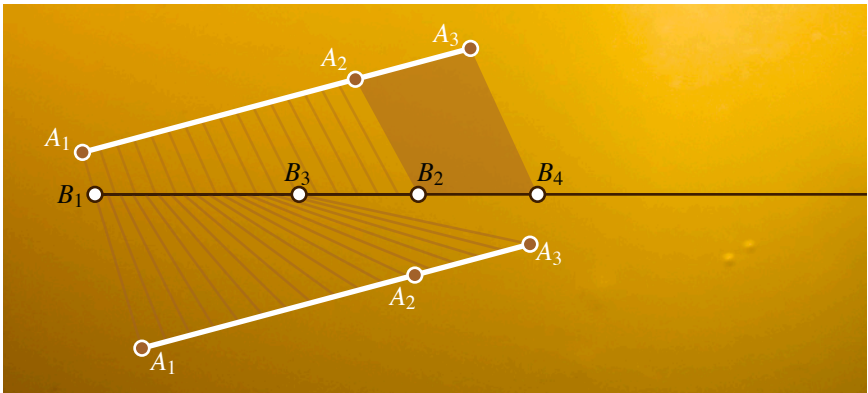
- (1)  $B_2 = B_3$    (2)  $B_1 * B_3 * B_2$    (3)  $B_1 * B_2 * B_3$ .

The last is what we want, so it is just a matter of ruling out the other two possibilities.

(1) Why can't  $B_3$  be equal to  $B_2$ ? With  $B_2 = B_3$ , both  $A_1A_2$  and  $A_1A_3$  are congruent to the same segment. Therefore they are two different constructions of a segment starting from  $A_1$  along  $A_1A_2 \rightarrow$  and congruent to  $B_1B_2$ . The Segment Construction Axiom says that there be only one.



*The case against case I*



The case against case II

(2) Why can't  $B_3$  be between  $B_1$  and  $B_2$ ? By the Segment Construction Axiom, there is a point  $B_4$  on the opposite side of  $B_2$  from  $B_1$  so that  $B_2B_4 \simeq A_2A_3$ . Now look:

$$B_1B_2 \simeq A_1A_2 \quad \& \quad B_2B_4 \simeq A_2A_3$$

so by the Segment Addition Axiom,  $B_1B_4 \simeq A_1A_3$ . This creates the same problem we ran into last time—two different segments  $B_1B_3$  and  $B_1B_4$ , both starting from  $B_1$  and going out along the same ray, yet both are supposed to be congruent to  $A_1A_3$ . □

## Triangle Congruence

Congruence of segments and angles is undefined, subject only to the axioms of congruence. But congruence of triangles *is* defined. It is defined in terms of the congruences of the segments and angles that make up the triangles.

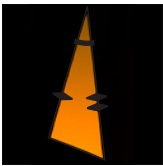
DEF: TRIANGLE CONGRUENCE

Two triangles  $\triangle ABC$  and  $\triangle A'B'C'$  are congruent if *all* of their corresponding sides and angles are congruent:

$$\begin{aligned} AB \simeq A'B' \quad BC \simeq B'C' \quad CA \simeq C'A' \\ \angle A \simeq \angle A' \quad \angle B \simeq \angle B' \quad \angle C \simeq \angle C'. \end{aligned}$$

Now that definition suggests that you have to match up six different things to say that two triangles are congruent. In actuality, triangles aren't really that flexible. Usually you only have to match up about half that many things. For example, the next result we will prove, the S·A·S Triangle Congruence Theorem, says that you only have to match up two sides of the triangles, and the angles between those sides, to show that the triangles are congruent. In this lesson, we begin the investigation of those minimum conditions.

Before we start studying these results, I would like to point out another way to view these theorems, this time in terms of *construction*. The triangle congruence theorems are set up to compare two triangles. Another way to think of them, though, is as a restriction on the way that a single triangle can be formed. To take an example, the S·A·S theorem below says that, modulo congruence, there is really only one triangle with a given pair of sides and a given intervening angle. Therefore, if you are building a triangle, and have decided upon two sides and an intervening angle, well, the triangle is decided— you don't get to choose the remaining side or the other two angles.



#### S·A·S TRIANGLE CONGRUENCE

In triangles  $\triangle ABC$  and  $\triangle A'B'C'$ , if

$$AB \simeq A'B' \quad \angle B \simeq \angle B' \quad BC \simeq B'C',$$

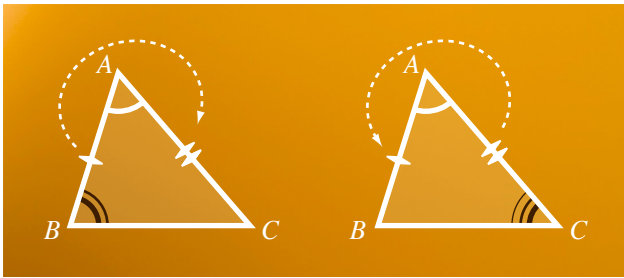
then  $\triangle ABC \simeq \triangle A'B'C'$ .

*Proof.* To show that two triangles are congruent, you have to show that three pairs of sides and three pairs of angles are congruent. Fortunately, two of the side congruences are given, and one of the angle congruences is given. The S·A·S axiom guarantees a second angle congruence,  $\angle A \simeq \angle A'$ . So that just leaves one angle congruence and one side congruence.

Let's do the angle first. You know, working abstractly creates a lot of challenges. On the few occasions when the abstraction makes things easier, it is a good idea to take advantage of it. This is one of those times. The S·A·S lemma tells us about  $\angle A$  in  $\triangle ABC$ . But let's not be misled by lettering. Because  $\triangle ABC = \triangle CBA$  and  $\triangle A'B'C' = \triangle C'B'A'$ , we can reorder the given congruences:

$$CB \simeq C'B' \quad \angle B \simeq \angle B' \quad BA \simeq B'A'.$$

Then the S·A·S lemma says that  $\angle C \simeq \angle C'$ . Sneaky isn't it? It is a completely legitimate use of the S·A·S axiom though.

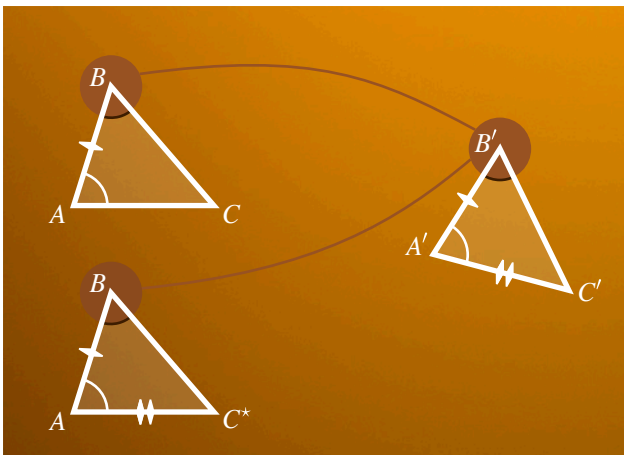


*Two orderings of the list of congruences for the SAS lemma.*

That just leaves the sides  $AC$  and  $A'C'$ . We are going to construct a congruent copy of  $\triangle A'B'C'$  on top of  $\triangle ABC$  (Euclid’s flawed proof of S·A·S in *The Elements* used a similar argument but without the axioms to back it up). Thanks to the Segment Construction Axiom, there is a unique point  $C^*$  on  $AC$  so that  $AC^* \simeq A'C'$ . Now if we can just show that  $C^* = C$  we will be done. Look:

$$BA \simeq B'A' \quad \angle A \simeq \angle A' \quad AC^* \simeq A'C'$$

By the S·A·S axiom then,  $\angle ABC^* \simeq \angle A'B'C'$ . That in turn means that  $\angle ABC^* \simeq \angle ABC$ . But wait— both of those angles are constructed on the same side of  $BA \rightarrow$ . According to the Angle Construction Axiom, that means they must be the same. That is,  $BC \rightarrow = BC^* \rightarrow$ . Both  $C$  and  $C^*$  are the intersection of this ray and the line  $AC$ . Since a ray can only intersect a line once,  $C$  and  $C^*$  do have to be the same.  $\square$



*To show the last sides are congruent, construct a third triangle from parts of the original two. The key to the location of  $C$  is the angle at  $B$ .*

One of the things that I really appreciate about the triangle congruence theorems is how transparent they are: their names tell us when to use them. For instance, you use S·A·S when you know congruences for two sides and the angle between them. And you use A·S·A when...



### A·S·A TRIANGLE CONGRUENCE

In triangles  $\triangle ABC$  and  $\triangle A'B'C'$ , if

$$\angle A \simeq \angle A' \quad AB \simeq A'B' \quad \angle B \simeq \angle B',$$

then  $\triangle ABC \simeq \triangle A'B'C'$ .

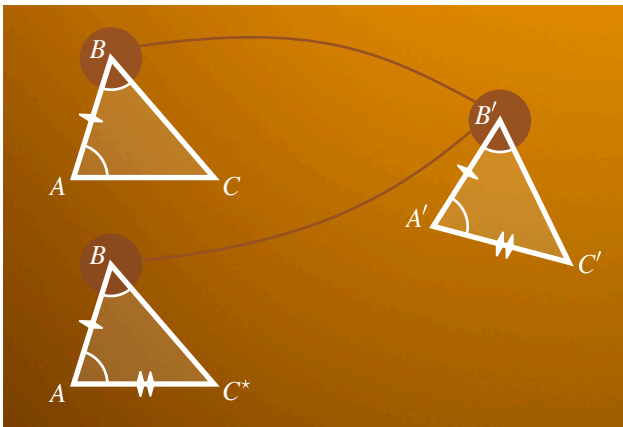
*Proof.* This time, it is a little easier—if we can just get one more side congruence, then S·A·S will provide the rest. You will probably notice some similarities between this argument and the last part of the S·A·S proof. Because of the Segment Construction Axiom, there is a point  $C^*$  on  $AC \rightarrow$  so that  $AC^* \simeq A'C'$ . Of course, the hope is that  $C^* = C$ , and that is what we need to show. To do that, observe that

$$BA \simeq B'A' \quad \angle A \simeq \angle A' \quad AC^* \simeq A'C'.$$

By S·A·S,  $\triangle ABC^* \simeq \triangle A'B'C'$ . In particular, look at what is happening at vertex  $B$ :

$$\angle ABC^* \simeq \angle A'B'C' \simeq \angle ABC.$$

There is only one way to make that angle on that side of  $BA \rightarrow$ , and that means  $BC^* \rightarrow = BC \rightarrow$ . Since both  $C$  and  $C^*$  are where this ray intersects  $\leftarrow AC \rightarrow$ ,  $C = C^*$ .



*Does this look familiar?*

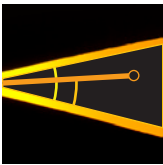


That’s the hard work. All that is left is to wrap up the argument. Since  $C = C^*$ ,  $AC = AC^*$ , and that means  $AC \simeq A'C'$ . Then

$$BA \simeq B'A' \quad \angle A \simeq \angle A' \quad AC \simeq A'C'$$

so by S·A·S,  $\triangle ABC \simeq \triangle A'B'C'$ . □

Let’s take a look at how the triangle congruence theorems can be put to work. This next theorem is the angle equivalent of the theorem at the start of this lesson relating congruence and the order of points.



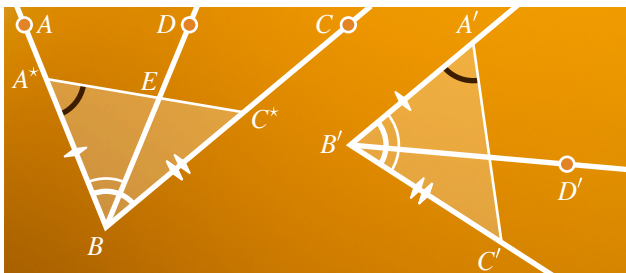
**THM: CONGRUENCE AND ANGLE INTERIORS**

Suppose that  $\angle ABC \simeq \angle A'B'C'$ . Suppose that  $D$  is in the interior of  $\angle ABC$ . And suppose that  $D'$  is located on the same side of  $\leftarrow AB \rightarrow$  as  $C$  so that  $\angle ABD \simeq \angle A'B'D'$ . Then  $D'$  is in the interior of  $\angle A'B'C'$ .

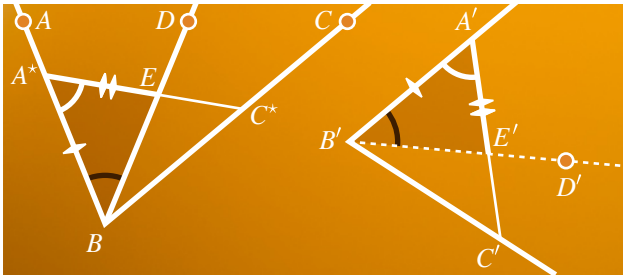
*Proof.* Because there is some flexibility in which points you choose to represent an angle, there is a good chance that our points are not organized in a very useful way. While we can’t change the rays or the angles themselves, we can choose other points to represent them. So the first step is to reposition our points in the most convenient way possible. Let  $A^*$  be the point on  $BA \rightarrow$  so that  $BA^* \simeq B'A'$ . Let  $C^*$  be the point on  $BC \rightarrow$  so that  $BC^* \simeq B'C'$ . Since  $D$  is in the interior of  $\angle ABC$ , the Crossbar Theorem guarantees that  $BD \rightarrow$  intersects  $A^*C^*$ . Let’s call this intersection  $E$ . Then

$$A^*B \simeq A'B' \quad \angle A^*BC^* \simeq \angle A'B'C' \quad BC^* \simeq B'C'$$

so by S·A·S,  $\triangle A^*BC^* \simeq \triangle A'B'C'$ .



*After repositioning points, the first use of SAS.*



The second use of SAS:  $E'$  and  $D'$  are on the same ray.

Okay, now let's turn our attention to the second configuration of points—the ones with the  $'$  marks. According to the Segment Construction Axiom, there is a point  $E'$  on  $A'C' \rightarrow$  so that  $A'E' \simeq A^*E$ . Furthermore, thanks to the earlier theorem relating congruence and order, since  $E$  is between  $A^*$  and  $C^*$ ,  $E'$  must be between  $A'$  and  $C'$ , and so it is in the interior of  $\angle A'B'C'$ . Now look:

$$BA^* \simeq B'A' \quad \angle BA^*E \simeq \angle B'A'E' \quad A^*E \simeq A'E'$$

so by S·A·S,  $\triangle BA^*E \simeq \triangle B'A'E'$ .

In particular, this means that  $\angle A^*BE \simeq \angle A'B'E'$ . But we were originally told that  $\angle A^*BE \simeq \angle A'B'D'$ . Since angle congruence is transitive this must mean that  $\angle A'B'D' \simeq \angle A'B'E'$ . Well, thanks to the Angle Construction Axiom, this means that the two rays  $B'D' \rightarrow$  and  $B'E' \rightarrow$  must be the same. Since  $E'$  is in the interior of  $\angle A'B'C'$ ,  $D'$  must be as well.  $\square$

## Symmetry in Triangles

I don't think it comes as a great surprise that in some triangles, two or even all three sides or angles may be congruent. Thanks to the triangle congruence theorems, we can show that these triangles are congruent to themselves in non-trivial ways. These non-trivial congruences reveal the internal symmetries of those triangles.

DEF: ISOSCELES, EQUILATERAL, SCALENE

If all three sides of a triangle are congruent, the triangle is *equilateral*. If exactly two sides of a triangle are congruent, the triangle is *isosceles*. If no pair of sides of the triangle is congruent, the triangle is *scalene*.

Here is one of those internal symmetry results. I put the others in the exercises.

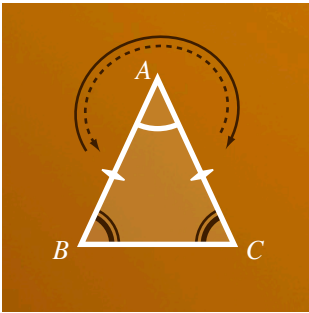
### THE ISOSCELES TRIANGLE THEOREM

In an isosceles triangle, the angles opposite the congruent sides are congruent.

*Proof.* Suppose  $\triangle ABC$  is isosceles, with  $AB \simeq AC$ . Then

$$AB \simeq AC \quad \angle A \simeq \angle A \quad AC \simeq AB,$$

so by S·A·S,  $\triangle ABC \simeq \triangle ACB$  (there's the non-trivial congruence of the triangle with itself). Comparing corresponding angles,  $\angle B \simeq \angle C$ .  $\square$



*Two orderings of the list of congruences for the SAS lemma.*

## Exercises

1. Given any point  $P$  and any segment  $AB$ , prove that there are infinitely many points  $Q$  so that  $PQ \simeq AB$ .
2. Verify that *triangle* congruence is an equivalent relation— that it is reflexive, symmetric, and transitive.
3. Prove the converse of the Isosceles Triangle Theorem: that if two interior angles of a triangle are congruent, then the sides opposite them must also be congruent.
4. Prove that all three interior angles of an equilateral triangle are congruent.
5. Prove that no two interior angles of a scalene triangle can be congruent.
6. In the exercises in Lesson 1, I introduced the Cartesian model and described how *point*, *line*, *on* and *between* are interpreted in that model. Let me extend that model now to include congruence. In the Cartesian model, segment congruence is defined in terms of the length of the segment, which, in turn, is defined using the distance function. If  $(x_a, y_a)$  and  $(x_b, y_b)$  are the coordinates of  $A$  and  $B$ , then the length of the segment  $AB$ , written  $|AB|$ , is

$$|AB| = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}.$$

Two segments are congruent if and only if they are the same length. With this interpretation, verify the first three axioms of congruence.

7. Angle congruence is the most difficult to interpret in the Cartesian model. Like segment congruence, angle congruence is defined via measure— in this case angle measure. You may remember from calculus that the dot product provides a way to measure the angle between two vectors: that for any two vectors  $v$  and  $w$ ,

$$v \cdot w = |v||w| \cos \theta,$$

where  $\theta$  is the angle between  $v$  and  $w$ . That is the key here. Given an angle  $\angle ABC$ , its measure, written  $(\angle ABC)$ , is computed as follows.

Let  $(x_a, y_a)$ ,  $(x_b, y_b)$  and  $(x_c, y_c)$  be the coordinates for points  $A$ ,  $B$ , and  $C$ , then define vectors

$$v = \langle x_a - x_b, y_a - y_b \rangle \quad w = \langle x_c - x_b, y_c - y_b \rangle.$$

and measure

$$(\angle ABC) = \cos^{-1} \left( \frac{v \cdot w}{|v||w|} \right).$$

Two angles are congruent if and only if they have the same angle measure. With this interpretation, verify the last three axioms of congruence.

