



**8. NARROWER AND WIDER
ANGLE COMPARISON**

These next two chapters are devoted to developing a measurement system for angles. It's really not that different from what we did in the last two chapters and again I would like to divide up the work so I don't feel like I am doing everything by myself. This time I will prove the results about the synthetic comparison of angles and I will let you prove the results which ultimately lead to the degree system of angle measurement.

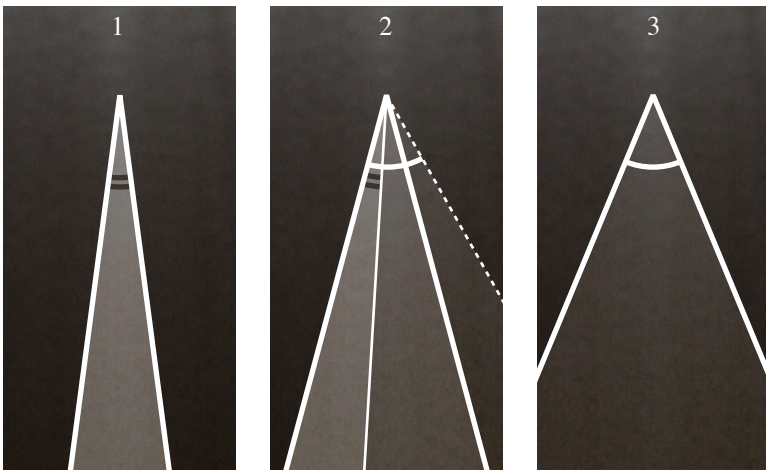
Synthetic angle comparison

The first step is to develop a way to compare angles so that you can look at two angles and say that one is smaller or larger than the other. I gave these definitions back in lesson 4, but in the interest of keeping everything together, and to introduce some notation, here they are again.

DEF: SMALLER AND LARGER ANGLES

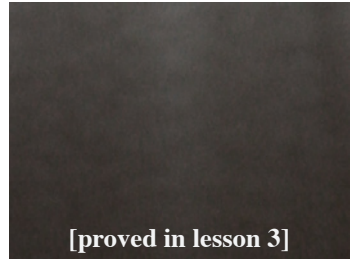
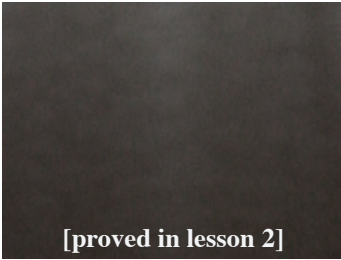
Given angles $\angle ABC$ and $\angle A'B'C'$, label C^* on the same side of AB as C so that $\angle ABC^* \simeq \angle A'B'C'$.

- \prec If C^* is in the interior of $\angle ABC$, then $\angle A'B'C'$ is *smaller than* $\angle ABC$, written $\angle A'B'C' \prec \angle ABC$.
- \succ If C^* is in the exterior of $\angle ABC$, then $\angle A'B'C'$ is *larger than* $\angle ABC$, written $\angle A'B'C' \succ \angle ABC$.



$$\angle 1 \prec \angle 2 \quad \angle 3 \succ \angle 2$$

In addition, the results of this section depend upon two results we proved a while ago.



THM: ORDERING RAYS

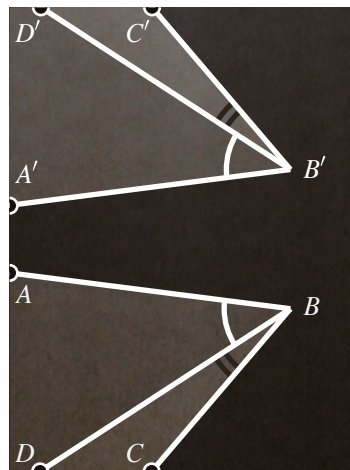
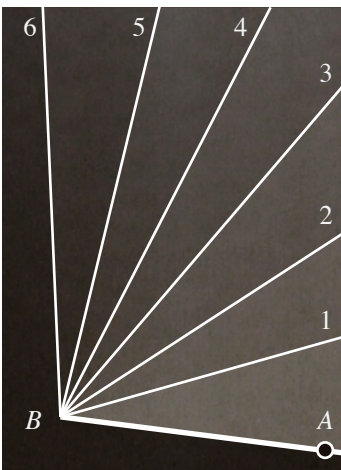
Given $n \geq 2$ rays with a common basepoint B which are all on the same side of the line $\leftarrow AB \rightarrow$ through B , there is an ordering of them:

$$r_1, r_2, \dots, r_n$$

so that if $i < j$ then r_i is in the interior of the angle formed by $BA \rightarrow$ and r_j .

THM: CONGRUENCE AND ANGLE INTERIORS

Given $\angle ABC \simeq \angle A'B'C'$ and that the point D is in the interior of $\angle ABC$. Suppose that D' is located on the same side of $\leftarrow A'B' \rightarrow$ as C so that $\angle ABD \simeq \angle A'B'D'$. Then D' is in the interior of $\angle A'B'C'$.



As with the segment comparison definitions, there is a potential issue with the definitions of \prec and \succ . What if we decided to construct C^* off of $BC \rightarrow$ instead of $BA \rightarrow$? Since $\angle ABC = \angle CBA$, and since we are interested in comparing the angles themselves, this notion of larger or smaller should not depend upon which ray we are building from. The next theorem tells us not to worry.

THM: \prec AND \succ ARE WELL DEFINED

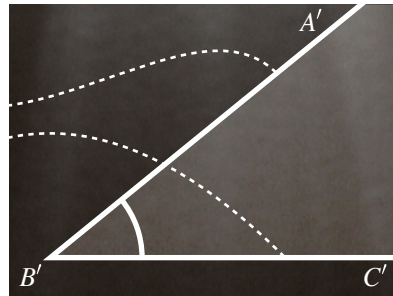
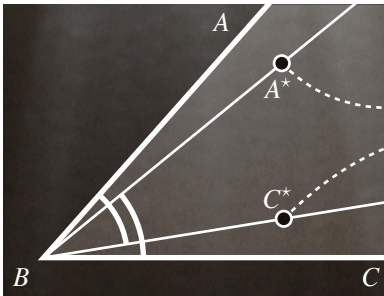
Given $\angle ABC$ and $\angle A'B'C'$, label:

C^* – a point on the same side of AB as C for which $\angle ABC^* \simeq \angle A'B'C'$

A^* – a point on the same side of BC as A for which $\angle CBA^* \simeq \angle A'B'C'$.

Then C^* is in the interior of $\angle ABC$ if and only if A^* is.

Proof. This is really a direct corollary of the “Congruence and Angle Interiors” result from lesson 3. You see, that is exactly what we have here: $\angle ABC \simeq \angle ABC$ and $\angle A^*BC \simeq \angle ABC^*$ and C^* is on the same side of AB as C , so if A^* is in the interior of $\angle ABC$, then C^* must be too. Conversely, A^* is on the same side of BC as A , so if C^* is in the interior, then A^* must be too. \square



When comparing angles, it doesn't matter which ray is used as the “base”.

Now let's take a look at some of the properties of synthetic angle comparison. I am focusing on the \prec version of these properties: the \succ version should be easy enough to figure out from these. There is nothing particularly elegant about these proofs. They mainly rely upon the two theorems listed above.

THM: TRANSITIVITY OF \prec

$\prec\prec$ If $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$ and $\angle A_2B_2C_2 \prec \angle A_3B_3C_3$,
then $\angle A_1B_1C_1 \prec \angle A_3B_3C_3$.

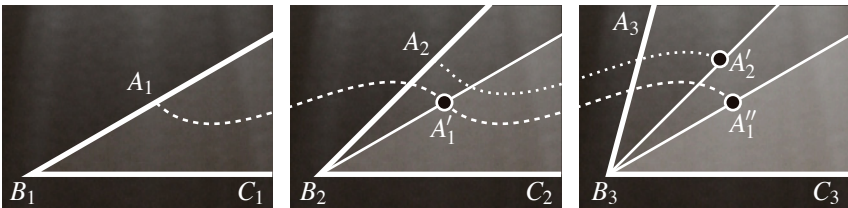
$\simeq\prec$ If $\angle A_1B_1C_1 \simeq \angle A_2B_2C_2$ and $\angle A_2B_2C_2 \prec \angle A_3B_3C_3$,
then $\angle A_1B_1C_1 \prec \angle A_3B_3C_3$.

$\prec\simeq$ If $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$ and $\angle A_2B_2C_2 \simeq \angle A_3B_3C_3$,
then $\angle A_1B_1C_1 \prec \angle A_3B_3C_3$.

Proof. Let me just take the first of these statements since the other two are easier. Most of the proof is just getting points shifted into a useful position.

1. Copy the first angle into the second: since $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$, there is a point A'_1 in the interior of $\angle A_2B_2C_2$ so that $\angle A_1B_1C_1 \simeq \angle A'_1B_2C_2$.
2. Copy the second angle in to the third: since $\angle A_2B_2C_2 \prec \angle A_3B_3C_3$, there is a point A'_2 in the interior of $\angle A_3B_3C_3$ so that $\angle A_2B_2C_2 \simeq \angle A'_2B_3C_3$.
3. Copy the first angle to the third (although we don't know quite as much about this one): pick a point A''_1 on the same side of B_3C_3 as A_1 so that $\angle A''_1B_3C_3 \simeq \angle A_1B_1C_1$.

Now we can get down to business. “Congruence and Angle Interiors”: since A'_1 is in the interior of $\angle A_2B_2C_2$, A''_1 has to be in the interior of $\angle A'_2B_3C_3$. “Ordering rays”: since $B_3A''_1 \rightarrow$ is in the interior of $\angle A_3B_3A'_2$, and since $B_3A'_2 \rightarrow$ is in the interior of $\angle A_3B_3C_3$, this means that $B_3A''_1 \rightarrow$ has to be in the interior of $\angle A_3B_3C_3$. Therefore $\angle A_1B_1C_1 \prec \angle A_3B_3C_3$. \square

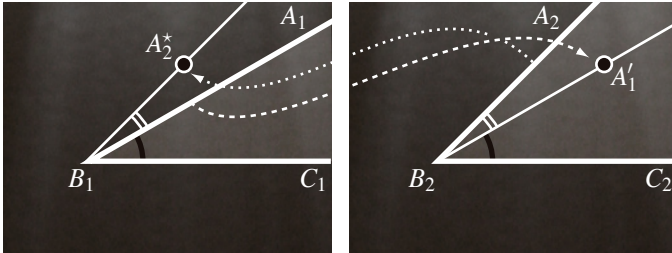


The transitivity of \prec .

THM: SYMMETRY BETWEEN \prec AND \succ

For any two angles $\angle A_1B_1C_1$ and $\angle A_2B_2C_2$, $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$ if and only if $\angle A_2B_2C_2 \succ \angle A_1B_1C_1$.

Proof. This is a direct consequence of the ‘‘Congruence and Angle Interiors’’ theorem. Suppose that $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$. Then there is a point A'_1 in the interior of $\angle A_2B_2C_2$ so that $\angle A_1B_1C_1 \simeq \angle A'_1B_2C_2$. Moving back to the first angle, there is a point A^*_2 on the opposite side of A_1B_1 from C_1 so that $\angle A_1B_1A^*_2 \simeq \angle A'_1B_2A_2$. By angle addition, $\angle A^*_2B_1C_1 \simeq \angle A_2B_2C_2$, and since A^*_2 is not in the interior of $\angle A_1B_1C_1$, that means $\angle A_2B_2C_2 \succ \angle A_1B_1C_1$. The other direction in this proof works very similarly so I won't go through it. \square



Relating \prec and \succ

THM: ORDERING FOUR RAYS

If A_2 and C_2 are in the interior of $\angle A_1BC_1$, then $\angle A_2BC_2 \prec \angle A_1BC_1$.

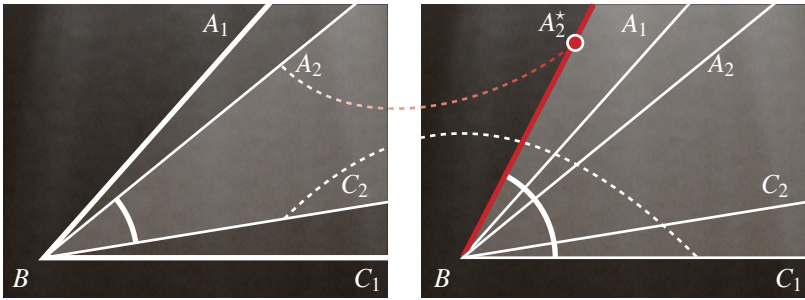
Proof. Locate A^*_2 on the same side of $\leftarrow BC_1 \rightarrow$ as A_1 so that

$$\angle A^*_2BC_1 \simeq \angle A_2BC_2.$$

Then the question is— is A^*_2 in the interior of $\angle A_1BC_1$? Well, let's suppose that it isn't. Then

$$\angle A_2BC_2 \prec \angle A^*_2BC_2 \prec \angle A^*_2BC_1.$$

Since we have established that \prec is transitive, that means $\angle A_2BC_2 \prec \angle A^*_2BC_1$. But this cannot be— those two angles are supposed to be congruent. Hence A^*_2 has to be in the interior of $\angle A_1BC_1$, and so $\angle A_2BC_2 \prec \angle A_1BC_1$. \square



Proof by contradiction of the “Ordering Four Rays” Theorem.

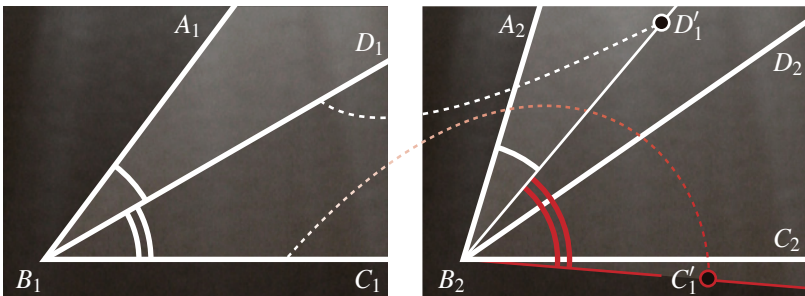
THM: ADDITIVITY OF \prec

Suppose that D_1 lies in the interior of $\angle A_1B_1C_1$ and that D_2 lies in the interior of $\angle A_2B_2C_2$. If $\angle A_1B_1D_1 \prec \angle A_2B_2D_2$ and $\angle D_1B_1C_1 \prec \angle D_2B_2C_2$, then $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$.

Proof. Find D'_1 in the interior of $\angle A_2B_2D_2$ so that $\angle A_2B_2D'_1 \simeq \angle A_1B_1D_1$. Find C'_1 on the opposite side of $\leftarrow B_2D'_1 \rightarrow$ from A_2 so that $\angle D'_1B_2C'_1 \simeq \angle D_1B_1C_1$. By angle addition, $\angle A_2B_2C'_1 \simeq \angle A_1B_1C_1$, so the question is whether or not C'_1 is in the interior of $\angle A_2B_2C_2$. Well, if it was not, then by the previous theorem

$$\angle D_2B_2C_2 \prec \angle D'_1B_2C'_1 \implies \angle D_2B_2C_2 \prec \angle D_1B_1C_1.$$

That is a contradiction (the angles were constructed to be congruent), so C'_1 will have to lie in the interior of $\angle A_2B_2C_2$, and so $\angle A_1B_1C_1 \prec \angle A_2B_2C_2$. □



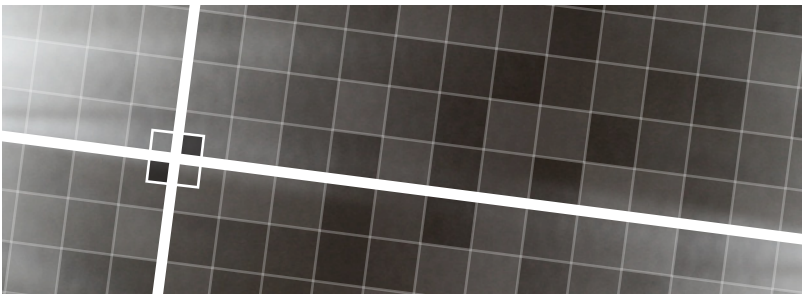
The proof by contradiction of the additivity of \prec

Right angles

Distance and segment length is based upon a completely arbitrary segment to determine unit length. Angle measure is handled differently— a specific angle is used as the baseline from which the rest is developed (although, at least in the degree measurement system, that angle is then assigned a pretty random measure). That angle is the right angle.

DEF: RIGHT ANGLE

A right angle is an angle which is congruent to its own supplement.



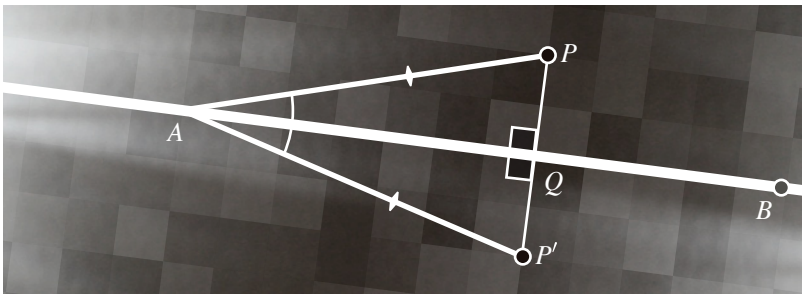
Right angles. In diagrams, squares angle markers are often used to indicate that an angle is right.

Now I didn't mention it at the time, but we have already stumbled across right angles once, in the proof of the S·S·S theorem. But it ought to be stated again, that

THM: RIGHT ANGLES, EXISTENCE

Right angles do exist.

Proof. We will prove that right angles exist by constructing one. Start with a segment AB . Now choose a point P which is not on the line $\leftarrow AB \rightarrow$. If $\angle PAB$ is congruent to its supplement, then it is a right angle, and that's it. If $\angle PAB$ is not congruent to its supplement (which is really a lot more likely), then there is a little more work to do. Thanks to the Segment and Angle Construction Axioms, there is a point P' on the opposite side of $\leftarrow AB \rightarrow$ from P so that $\angle P'AB \simeq \angle PAB$ (angle construction) and $AP' \simeq AP$



Proof of existence by construction.

(segment construction). Since P and P' are on opposite sides of \overleftrightarrow{AB} , the segment PP' has to intersect the line \overleftrightarrow{AB} . Call that point of intersection Q . With that construction,

$$PA \simeq P'A \quad \angle PAQ \simeq \angle P'AQ \quad AQ \simeq AQ$$

so by S·A·S triangle congruence theorem, $\triangle PAQ \simeq \triangle P'AQ$. Out of those two triangles, the relevant congruence is between the two angles that share the vertex Q : $\angle AQP \simeq \angle AQP'$. These angles are supplements. They are congruent. By definition, they are right angles. \square

Okay, so they are out there. But how many are there? The next result is something like a uniqueness statement– that there is really only one right angle “modulo congruence”.

THM: RIGHT ANGLES AND CONGRUENCE

Suppose that $\angle ABC$ is a right angle. Then $\angle A'B'C'$ is a right angle if and only if it is congruent to $\angle ABC$.

Proof. This is an “if and only if” statement, and that means that there are two directions to prove.

\implies If $\angle A'B'C'$ is a right angle, then $\angle A'B'C' \simeq \angle ABC$.

\impliedby If $\angle A'B'C' \simeq \angle ABC$, then $\angle A'B'C'$ is a right angle.

\implies To start, let's go ahead and mark a few more points so that we can refer to the supplements of these angles. Mark the points

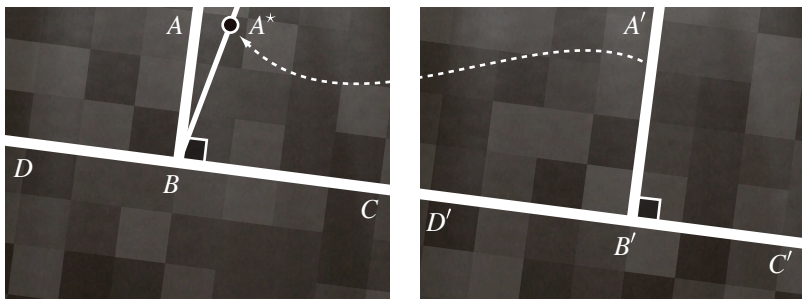
D on $\leftarrow BC \rightarrow$ so that $D * B * C$ and

D' on $\leftarrow B'C' \rightarrow$ so that $D' * B' * C'$.

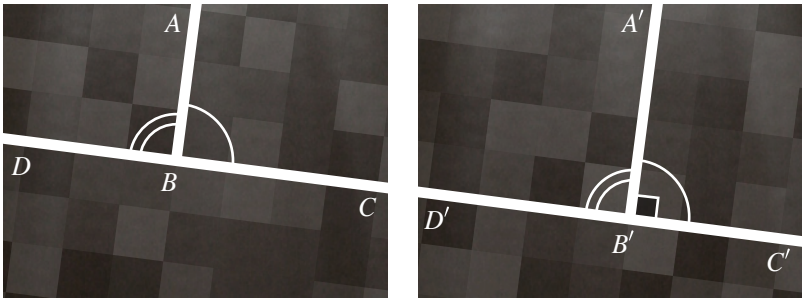
Therefore $\angle ABC$ and $\angle ABD$ are a supplementary pair, as are $\angle A'B'C'$ and $\angle A'B'D'$. Now suppose that both $\angle ABC$ and $\angle A'B'C'$ are right angles. Thanks to the Angle Construction Axiom, it is possible to build a congruent copy of $\angle A'B'C'$ on top of $\angle ABC$: there is a ray $BA^* \rightarrow$ on the same side of BC as A so that $\angle A^*BC \simeq \angle A'B'C'$. Earlier we proved that the supplements of congruent angles are congruent, so that means $\angle A^*BD \simeq \angle A'B'D'$. How, though, does $\angle A^*BC$ compare to $\angle ABC$? If $BA^* \rightarrow$ and $BA \rightarrow$ are the same ray, then the angles are equal, meaning that $\angle ABC$ and $\angle A'B'C'$ are congruent— which is what we want. But what happens if the two rays are not equal? In that case one of two things can happen: either $BA^* \rightarrow$ is in the interior of $\angle ABC$, or it is in the interior of $\angle ABD$. Both of these cases are going to lead to essentially the same problem, so let me just focus on the first one. In that case, A^* is in the interior of $\angle ABC$, so $\angle A^*BC \prec \angle ABC$, but A^* is in the exterior of $\angle ABD$, so $\angle A^*BD \succ \angle ABD$. That leads to a string of congruences and inequalities:

$$\angle A'B'C' \simeq \angle A^*BC \prec \angle ABC \simeq \angle ABD \prec \angle A^*BD \simeq \angle A'B'D'.$$

Because of the transitivity of \prec then, $\angle A'B'C' \prec \angle A'B'D'$. This can't be— those two supplements are supposed to be congruent. The second scenario plays out in the same way, with \succ in place of \prec . Therefore $BA^* \rightarrow$ and $BA \rightarrow$ have to be the same ray, and so $\angle A'B'C' \simeq \angle ABC$.



Any two right angles are congruent: if one right angle were larger or smaller than another, it could not be congruent to its complement.



If an angle is congruent to a right angle, it is a right angle too.

\Leftarrow The other direction is easier. Suppose that $\angle A'B'C' \simeq \angle ABC$ and that $\angle ABC$ is a right angle. Let's recycle the points D and D' from the first part of the proof. The angles $\angle A'B'D'$ and $\angle ABD$ are supplementary to congruent angles, so they too must be congruent. Therefore

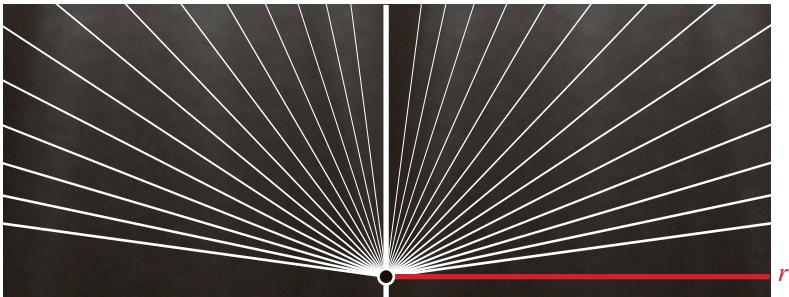
$$\angle A'B'C' \simeq \angle ABC \simeq \angle ABD \simeq \angle A'B'D'$$

and so we can see that $\angle A'B'C'$ is congruent to its supplement— it must be a right angle. □

With \prec and \succ and with right angles as a point of comparison, we have a way to classify non-right angles.

DEF: ACUTE AND OBTUSE

An angle is *acute* if it is smaller than a right angle. An angle is *obtuse* if it is larger than a right angle.



Rays that form an obtuse angle with r .

Rays that form an acute angle with r .

Exercises

1. Verify that the supplement of an acute angle is an obtuse angle and that the supplement of an obtuse angle is an acute angle.
2. Prove that an acute angle cannot be congruent to an obtuse angle (and vice versa).
3. Two intersecting lines are *perpendicular* if the angles formed at their intersection are right angles. For any line ℓ and point P , prove that there is a unique line through P which is perpendicular to ℓ . Note that there are two scenarios: P may or may not be on ℓ .
4. Consider two isosceles triangles with a common side: $\triangle ABC$ and $\triangle A'BC$ with $AB \simeq AC$ and $A'B \simeq A'C$. Prove that $\leftarrow AA' \rightarrow$ is perpendicular to $\leftarrow BC \rightarrow$.
5. Two angles are *complementary* if together they form a right angle. That is, if D is in the interior of a right angle $\angle ABC$, then $\angle ABD$ and $\angle DBC$ are complementary angles. Prove that every acute angle has a complement. Prove that if $\angle ABC$ and $\angle A'B'C'$ are congruent acute angles, then their complements are also congruent.
6. Verify that if ℓ_1 is perpendicular to ℓ_2 and ℓ_2 is perpendicular to ℓ_3 , then either $\ell_1 = \ell_3$, or ℓ_1 and ℓ_3 are parallel.

