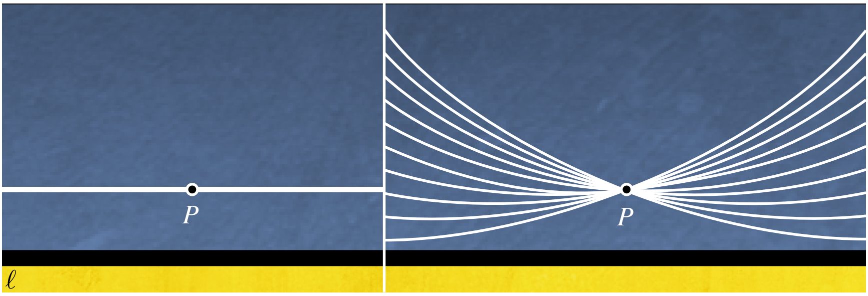
2 EUCLIDEAN GEOMETRY

My goal with all of these lessons is to provide an introduction to both Euclidean non-Euclidean geometry. The two geometries share many features, but they also have very fundamental and radical differences. Neutral geometry is the part of the path they have in common and that is what we have been studying so far, but I think we have finally come to the fork in the path. That fork comes when you try to answer this question:

> Given a line ℓ and a point P which is not on ℓ, how many lines pass through P and are parallel to ℓ?

Using just the axioms of neutral geometry, you can prove that there is always at least one such parallel. You can also prove that if there is more than one parallel, then there must be infinitely many. But that is the extent of what the neutral axioms can say. The neutral axioms just aren't enough to determine whether there is one parallel or many. This is what separates Euclidean and non-Euclidean geometry– a single axiom: the final axiom of Euclidean geometry calls for *exactly one* parallel, the final axiom of non-Euclidean geometry calls for *more than one* parallel.

13 REGARDING PARALLELS, **A DECISION IS MADE**



Euclidean parallel *non-Euclidean parallels*

The next several lessons are devoted to Euclidean geometry. Now you have to remember that Euclidean geometry is several millenia old, so there is a lot of it. All that I hope to do in these lessons is to cover the fundamentals, but there are many excellent books that do much more. *Geometry Revisited* [1] by Coxeter and Greitzer is an excellent one.

The first order of business is to put that final axiom in place. There are many formulations of the parallel axiom for Euclidean geometry, but the one that I think gets right to the heart of the matter is Playfair's Axiom, named after the Scottish mathematician John Playfair.

PLAYFAIR'S AXIOM

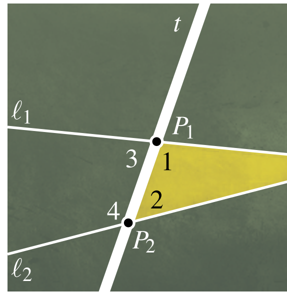
Let ℓ be a line, and let P be a point which is not on ℓ. Then there is exactly one line through P which is parallel to ℓ.

In this lesson I would like to look at a small collection of theorems which are almost immediate consequences of this axiom, and as such, are at the very core of Euclidean geometry. The first of these is Euclid's Fifth Postulate. This is the controversial postulate in *The Elements*, but also the one that guarantees the same parallel behavior that Playfair's Axiom provides. In my opinion, Euclid's postulate is a little unwieldy, particularly when compared to Playfair's Axiom, but it is the historical impetus for so much of what followed. So let's use Playfair's Axiom to prove Euclid's Fifth Postulate.

EUCLID'S FIFTH POSTULATE

If lines ℓ_1 and ℓ_2 are crossed by a transversal t, and the sum of adjacent interior angles on one side of t measure less than $180°$, then ℓ_1 and ℓ_2 intersect on that side of t.

Proof. First, some labels. Start with lines ℓ_1 and ℓ_2 crossed by transversal t. Label P_1 and P_2, the points of intersection of t with ℓ_1 and ℓ_2 respectively. On one side of t, the two adjacent interior angles should add up to less than $180°$. Label the one at P_1 as $\angle 1$ and the one at P_2 at $\angle 2$. Label the supplement of $\angle 1$ as $\angle 3$ and label the supplement of $\angle 2$ as $\angle 4$.

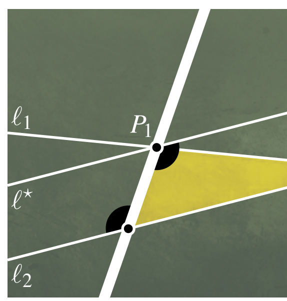


The labels.

Primarily, of course, this postulate is about the location of the intersection of ℓ_1 and ℓ_2. But you don't want to overlook an important prerequisite: the postulate is also guaranteeing that ℓ_1 and ℓ_2 *do* intersect. That's really the first thing we need to show. Note that $\angle 1$ and $\angle 4$ are alternate interior angles, but they are not congruent– if they were, their supplements $\angle 2$ and $\angle 3$ would be too, and then

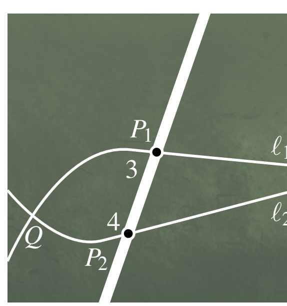
$$(\angle 1) + (\angle 2) = (\angle 1) + (\angle 3) = 180°.$$

There is, however, another line $\ell^\star$ through P_1 which does form an angle congruent to $\angle 4$ (because of the Angle Construction Postulate), and by the Alternate Interior Angle Theorem, $\ell^\star$ must be parallel to ℓ_2. Because of Playfair's Axiom, $\ell^\star$ is the only parallel to ℓ_2 through P_1. That means ℓ_1 intersects ℓ_2.

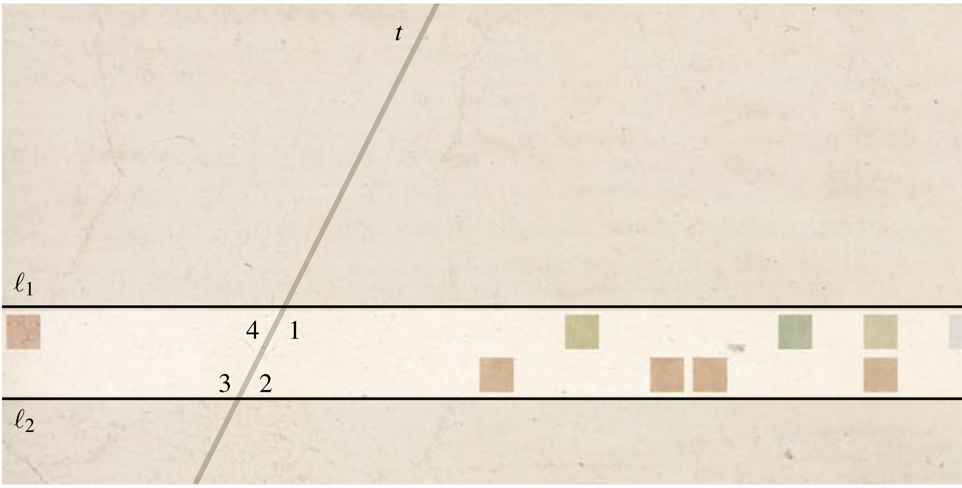


Constructing the unique parallel.

The second part of the proof is to figure out on which side of t that ℓ_1 and ℓ_2 cross. Let's see what would happen if they intersected at a point Q on the wrong side of t: the side with $\angle 3$ and $\angle 4$. Then the triangle $\triangle P_1P_2Q$ would have two interior angles, $\angle 3$ and $\angle 4$, which add up to more than $180°$. This violates the Saccheri-Legendre theorem. So ℓ_1 and ℓ_2 cannot intersect on the side of t with $\angle 3$ and $\angle 4$ and that means that they must intersect on the side with $\angle 1$ and $\angle 2$. $\square$



An impossible triangle on the wrong side of t.



One of the truly useful theorems of neutral geometry is the Alternate Interior Angle Theorem. In fact, we just used it in the last proof. But you may recall from high school geometry, that the converse of that theorem is often even more useful. The problem is that the converse of the Alternate Interior Angle Theorem can't be proved using just the axioms of neutral geometry. It depends upon Euclidean behavior of parallel lines.

CONVERSE OF THE ALTERNATE INTERIOR ANGLE THEOREM
If ℓ_1 and ℓ_2 are parallel, then the pairs of alternate interior angles formed by a transversal t are congruent.

Proof. Consider two parallel lines crossed by a transversal. Label adjacent interior angles: $\angle 1$ and $\angle 2$, and $\angle 3$ and $\angle 4$, so that $\angle 1$ and $\angle 4$ are supplementary and $\angle 2$ and $\angle 3$ are supplementary. That means that the pairs of alternate interior angles are $\angle 1$ and $\angle 3$ and $\angle 2$ and $\angle 4$. Now, we just have to do a little arithmetic. From the two pairs of supplementary angles:

$$\begin{cases} (\angle 1) + (\angle 4) = 180^\circ & (i) \\ (\angle 2) + (\angle 3) = 180^\circ. & (ii) \end{cases}$$

Notice that if you add all four angles together, then

$$(\angle 1) + (\angle 2) + (\angle 3) + (\angle 4) = 360^\circ.$$

Now, here is where Euclid's Fifth comes into play– and actually, we will need to use the contrapositive. You see, ℓ_1 and ℓ_2 are parallel, and that means that they do not intersect on either side of t. Therefore Euclid's Fifth says that on neither side of t may the sum of adjacent interior angles be less than 180°:

$$\begin{cases} (\angle 1) + (\angle 2) \geq 180^\circ \\ (\angle 3) + (\angle 4) \geq 180^\circ. \end{cases}$$

If either one of these sums was greater than 180°, though, the sum of all four angles would have to be more than 360°– we saw above that is not the case, so the inequalities are actually equalities:

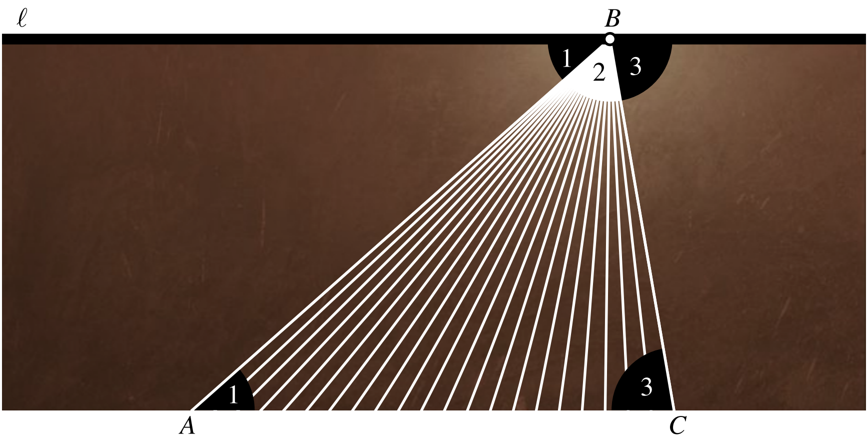
$$\begin{cases} (\angle 1) + (\angle 2) = 180^\circ & (iii) \\ (\angle 3) + (\angle 4) = 180^\circ. & (iv) \end{cases}$$

Now you have two systems of equations with four unknowns– it is basic algebra from here. Subtract equation (iv) from equation (i) to get $(\angle 1) = (\angle 3)$. Subtract equation (iii) from equation (i) to get $(\angle 2) = (\angle 4)$. The alternate interior angles are congruent. $\square$

One of the key theorems we proved in the neutral geometry section was the Saccheri-Legendre Theorem: that the angle sum of a triangle is at most 180°. That's all you can say with the axioms of neutral geometry, but in a world with Playfair's Axiom and the converse of the Alterante Interior Angle Theorem, there can be only one triangle angle sum.

THM
The angle sum of a triangle is 180°.

Proof. Consider a triangle $\triangle ABC$. By Playfair's Axiom, there is a unique line ℓ through B which is parallel to $\leftarrow AC \rightarrow$. That line and the rays $BA\rightarrow$ and $BC\rightarrow$ form three angles, $\angle 1$, $\angle 2$ and $\angle 3$ as I have shown in the illustration below.



By the converse of the Alternate Interior Angle Theorem, two pairs of alternate interior angles are congruent:

$$\angle 1 \simeq \angle A \quad \angle 3 \simeq \angle C.$$

Therefore, the angle sum of $\triangle ABC$ is

$$\begin{aligned} s(\triangle ABC) &= (\angle A) + (\angle B) + (\angle C) \\ &= (\angle 1) + (\angle 2) + (\angle 3) \\ &= 180^\circ. \end{aligned}$$

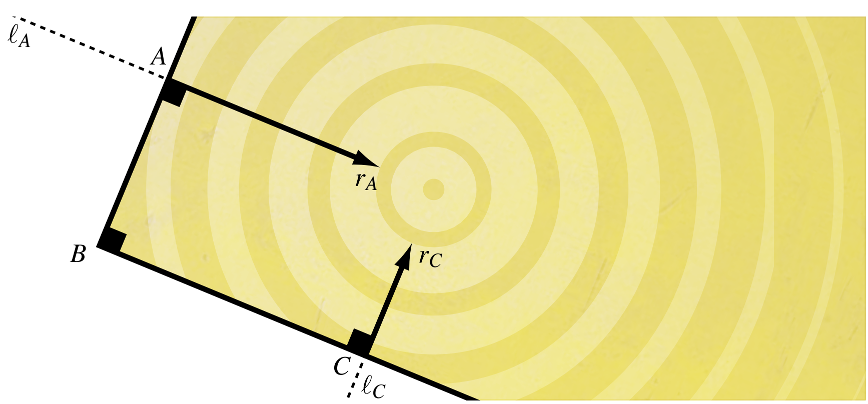
$\square$

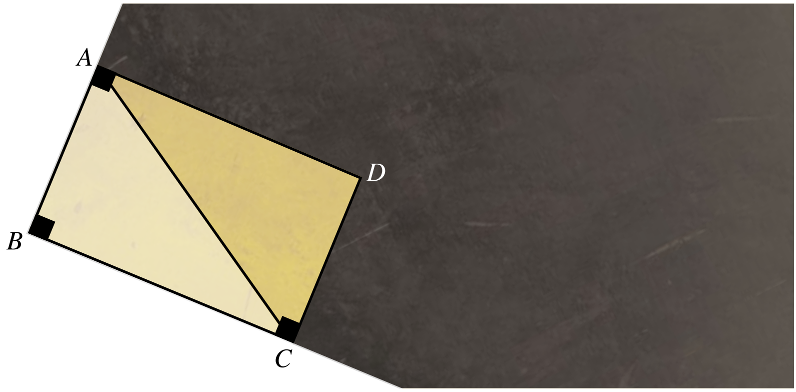
In the last lesson on quadrilaterals I talked a little bit about the uncertain status of rectangles in neutral geometry– that it is pretty easy to make a convex quadrilateral with three right angles, but that once you have done that, there is no guarantee that the fourth angle will be a right angle. Here it is now in the Euclidean context:

RECTANGLES EXIST

Let $\angle ABC$ be a right angle. Let r_A and r_B be rays so that: r_A has endpoint A, is on the same side of $\leftarrow AB \rightarrow$ as C, and is perpendicular to $\leftarrow AB \rightarrow$; r_C has endpoint C, is on the same side of $\leftarrow BC \rightarrow$ as A, and is perpendicular to $\leftarrow BC \rightarrow$. Then r_A and r_C intersect at a point D, and the angle fomed at this intersection, $\angle ADC$, is a right angle. Therefore $\square ABCD$ is a rectangle.

Proof. The first bit of business is to make sure that r_A and r_C intersect. Let ℓ_A and ℓ_C be the lines containing r_A and r_C respectively. By the Alternate Interior Angle Theorem, the right angles at A and B mean that ℓ_A and $\leftarrow BC \rightarrow$ are parallel. So $\leftarrow BC \rightarrow$ is the one line parallel to ℓ_A through C, and that means that ℓ_C cannot be parallel to ℓ_A: it has to intersect ℓ_A. Let's call that point of intersection D. Now in the statement of the theorem, I claim that it is the *rays*, not the lines, that intersect. That means that we need to rule out the possibility that the intersection of ℓ_A and ℓ_C might happen on one (or both) of the opposite rays. Observe that since ℓ_A is parallel to $\leftarrow BC \rightarrow$, all of the points of ℓ_A are on the same side of $\leftarrow BC \rightarrow$ as A. None of the points of r_C^{op} are on that side of BC, so D cannot be on r_C^{op}. Likewise, all the points of ℓ_C are on the same side of $\leftarrow AB \rightarrow$ as C. None of the points of r_A^{op} are on that side of AB, so D cannot be on r_A^{op}.





So now we have a quadrilateral $\square ABCD$ with three right angles, $\angle A$, $\angle B$, and $\angle C$. It is actually a convex quadrilateral too (I leave it to you to figure out why), so the diagonal AC divides $\square ABCD$ into two triangles $\triangle ABC$ and $\triangle ADC$. Then, since the angle sum of a triangle is 180°,

$$\begin{aligned} s(\triangle ABC) + s(\triangle ADC) &= 180^\circ + 180^\circ \\ (\angle CAB) + (\angle B) + (\angle ACB) + (\angle CAD) + (\angle D) + (\angle ACD) &= 360^\circ \\ (\angle A) + (\angle B) + (\angle C) + (\angle D) &= 360^\circ \\ 90^\circ + 90^\circ + 90^\circ + (\angle D) &= 360^\circ \\ (\angle D) &= 90^\circ. \end{aligned}$$

□

That means that, yes, rectangles *do* exist in Euclidean geometry. In the next lemma, I have listed some basic properties of a rectangle. I will leave it to you to prove these (they aren't hard).

LEM: PROPERTIES OF RECTANGLES

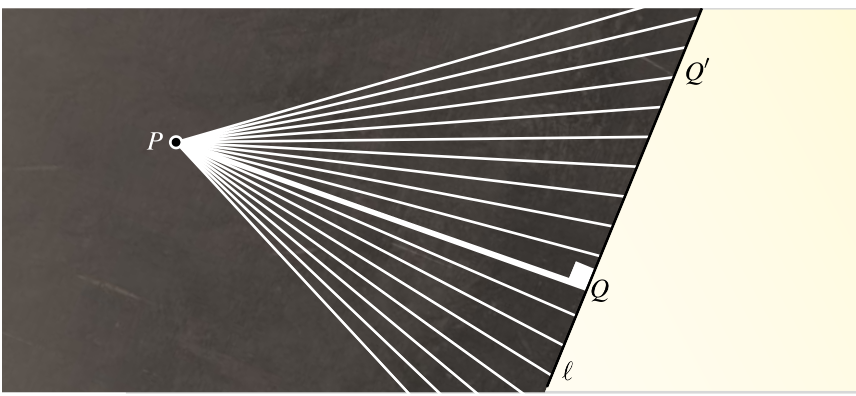
Let $\square ABCD$ be a rectangle. Then

1. $\leftarrow AB \rightarrow$ is parallel to $\leftarrow CD \rightarrow$ and $\leftarrow AD \rightarrow$ is parallel to $\leftarrow BC \rightarrow$
2. $AB \simeq CD$ and $AD \simeq BC$ and $AC \simeq BD$.

For the last result of this section, I would like to get back to parallel lines. One of the things that we will see when we study non-Euclidean geometry is that parallel lines tend to diverge from each other. That doesn't happen in non-Euclidean geometry. It is one of the key differences between the two geometries. Let me make this more precise. Suppose that P is a point which is not on a line ℓ. Define the distance from P to ℓ to be the minimum distance from P to a point on ℓ:

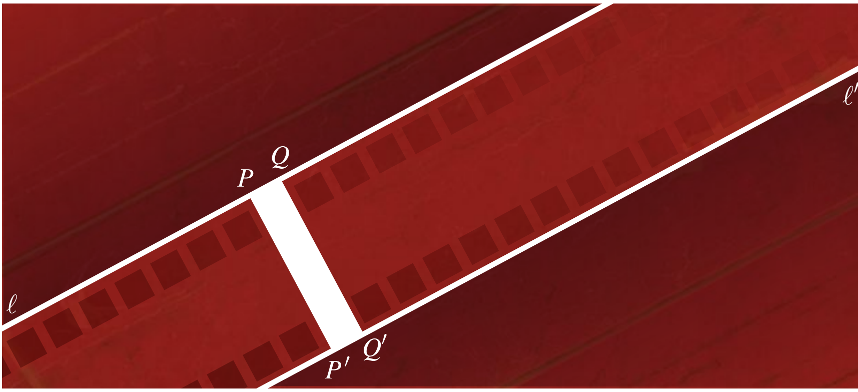
$$d(P,\ell) = \min\left\{ |PQ| \,\middle|\, Q \text{ is on } \ell \right\}.$$

That minimum actually occurs when Q is the foot of the perpendicular to ℓ through P. To see why, let Q' be any other point on ℓ. In $\triangle PQQ'$, the right angle at Q is the largest angle. By the Scalene Triangle Theorem, that means that the opposite side PQ' has to be the longest side, and so $|PQ'| > |PQ|$.



The distance from a point to a line is measured along the segment from the point to the line which is perpendicular to the line.

Now, for a given pair of parallel lines, that distance as measured along perpendiculars does not change.



THM: PARALLEL LINES ARE EVERYWHERE EQUIDISTANT

If ℓ and ℓ' are parallel lines, then the distance from a point on ℓ to ℓ' is constant. In other words, if P and Q are points on ℓ, then

$$d(P,\ell') = d(Q,\ell').$$

Proof. Let P' and Q' be the feet of the perpendiculars on ℓ' from P and Q respectively. That way,

$$d(P,\ell') = |PP'| \quad d(Q,\ell') = |QQ'|.$$

Then $\angle PP'Q'$ and $\angle QQ'P'$ are right angles. By the converse of the Alternate Interior Angle Theorem, $\angle P$ and $\angle Q$ are right angles too– so $\square PQQ'P'$ is a rectangle. Using the previous lemma on rectangles, PP' and QQ', which are the opposite sides of a rectangle, are congruent. $\square$

Exercises

1. Suppose that ℓ_1, ℓ_2 and ℓ_3 are three distinct lines such that: ℓ_1 and ℓ_2 are parallel, and ℓ_2 and ℓ_3 are parallel. Prove then that ℓ_1 and ℓ_3 are parallel.

2. Find the angle sum of a convex n-gon as a function of n.

3. Prove that the opposite sides and the opposite angles of a parallelogram are congruent.

4. Consider a convex quadrilateral $\square ABCD$. Prove that the two diagonals of $\square ABCD$ bisect each other if and only if $\square ABCD$ is a parallelogram.

5. Show that a parallelogram $\square ABCD$ is a rectangle if and only if $AC \simeq BD$.

6. Suppose that the diagonals of a convex quadrilateral $\square ABCD$ intersect one another at a point P and that

$$AP \simeq BP \simeq CP \simeq DP.$$

   Prove that $\square ABCD$ is a rectangle.

7. Suppose that the diagonals of a convex quadilateral bisect one another at right angles. Prove that the quadrilateral must be a rhombus.

8. Consider a triangle $\triangle ABC$ and three additional points A', B' and C'. Prove that if AA', BB' and CC' are all congruent and parallel to one another then $\triangle ABC \simeq \triangle A'B'C'$.

9. Verify that the Cartesian model (as developed through the exercises in lessons 1 and 3) satisfies Playfair's Axiom.

References

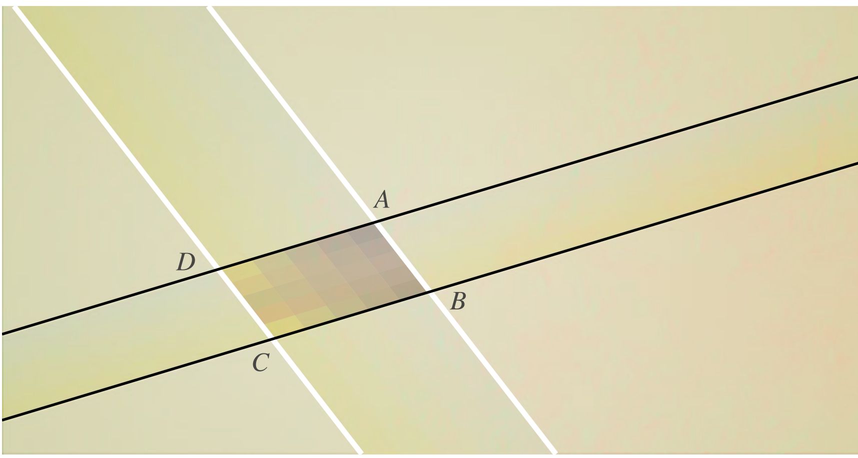
[1] H.S.M. Coxeter and Samuel L. Greitzer. *Geometry Revisited*. Random House, New York, 1st edition, 1967.

14 PARALLEL PROJECTION

Some calisthenics to start the lesson

In the course of this lesson, we are going to need to use a few facts dealing with parallelograms. First, let me remind of the proper definition of a parallelogram.

DEF: PARALLELOGRAM
A *parallelogram* is a simple quadrilateral whose opposite sides are parallel.



Now on to the facts about parallelograms that we will need for this lesson. None of their proofs are that difficult, but they would be a good warm-up for this lesson.

1 Prove that in a parallelogram, the two pairs of opposite sides are congruent and the two pairs of opposite angles are congruent.

2 Prove that if a convex quadrilateral has one pair of opposite sides which are both parallel and congruent, then it is a parallelogram.

3 Let $\square ABB'A'$ be a simple quadrilateral. Verify that if AA' and BB' are parallel, but AB and $A'B'$ are not, then AA' and BB' cannot be congruent.

Parallel projection

The purpose of this lesson is to introduce a mechanism called parallel projection, a particular kind of mapping from points on one line to points on another. Parallel projection is the piece of machinery that you have to have in place to really understand similarity, which is in turn essential for so much of what we will be doing in the next lessons. The primary goal of this lesson is to understand how distances between points may be distorted by the parallel projection mapping. Once that is figured out, we will be able to turn our attention to the geometry of similarity.

DEF: PARALLEL PROJECTION

A *parallel projection* from one line ℓ to another ℓ' is a map Φ which assigns to each point P on ℓ a point $\Phi(P)$ on ℓ' so that all the lines connecting a point and its image are parallel to one another.

It is easy to construct parallel projections. Any one point P on ℓ and its image $\Phi(P)$ on ℓ' completely determines the projection: for any other point Q on ℓ there is a unique line which passes through Q and is parallel to the line $\leftarrow P\Phi(P) \rightarrow$. Wherever this line intersects ℓ' will have to be $\Phi(Q)$. There are only two scenarios where this construction will not work out: (1) if P is the intersection of ℓ and ℓ', then the lines of projection run parallel to ℓ' and so fail to provide a point of intersection; and (2) if $\Phi(P)$ is the intersection of ℓ and ℓ', then the lines of projection actually coincide rather than being parallel.

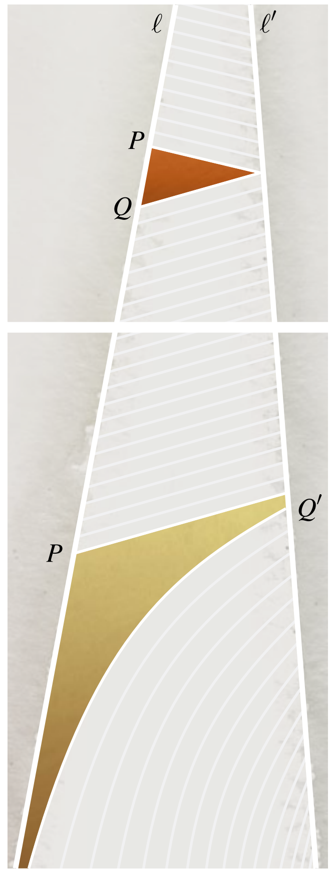
The path from a point P on ℓ to a point P' on ℓ' defines a parallel projection as long as neither P nor P' is the intersection of ℓ and ℓ' (as shown at right).

THM: PARALLEL PROJECTION IS A BIJECTION

A parallel projection is both one-to-one and onto.

Proof. Consider a parallel projection $\Phi : \ell \to \ell'$. First let's see why Φ is one-to-one. Suppose that it is not. That is, suppose that P and Q are two distinct points on ℓ but that $\Phi(P) = \Phi(Q)$. Then the two projecting lines $\leftarrow P\Phi(P) \rightarrow$ and $\leftarrow Q\Phi(Q) \rightarrow$, which ought to be parallel, actually share a point. This can't happen.

Now let's see why Φ is onto, so take a point Q' on ℓ'. We need to make sure that there is a point Q on ℓ so that $\Phi(Q) = Q'$. To get a sense of how Φ is casting points from ℓ to ℓ', let's consider a point P on ℓ and its image $\Phi(P)$ on ℓ'. The projecting line that should lead from Q to Q' ought to be parallel to $\leftarrow P\Phi(P) \rightarrow$. Now, there is a line which passes through Q' and is parallel to $\leftarrow P\Phi(P) \rightarrow$. The only question, then, is whether that line intersects ℓ– if it does, then we have found our Q. What if it doesn't though? In that case, our line is parallel to both $\leftarrow P\Phi(P) \rightarrow$ and ℓ. That would mean that $\leftarrow P\Phi(P) \rightarrow$ and ℓ are themselves parallel. Since P is on both of these lines, we know that cannot be the case. □



Since parallel projection is a bijection, I would like to use a naming convention for the rest of this lesson that I think makes things a little more readable. I will use a prime mark $'$ to indicate the parallel projection of a point. So $\Phi(P) = P'$, $\Phi(Q) = Q'$, and so on.

Parallel projection, order, and congruence.

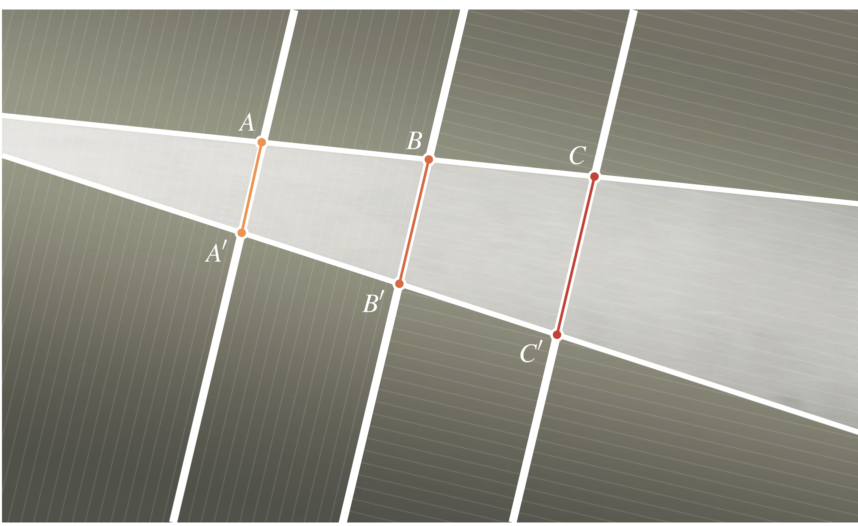
So far we have seen that parallel projection establishes a correspondence between the points of one line and the points of another. What about the order of those points? Can points get shuffled up in the process of a parallel projection? Well, ... no.

THM: PARALLEL PROJECTION AND ORDER
Let $\Phi : \ell \to \ell'$ be a parallel projection. If A, B, and C are points on ℓ and B is between A and C, then B' is between A' and C'.

Proof. Because B is between A and C, A and C must be on opposite sides of the line $\leftarrow BB' \rightarrow$. But:

$\leftarrow AA' \rightarrow$ does not intersect $\leftarrow BB' \rightarrow$ so A' has to be on the same side of $\leftarrow BB' \rightarrow$ as A.

$\leftarrow CC' \rightarrow$ does not intersect $\leftarrow BB' \rightarrow$ so C' has to be on the same side of $\leftarrow BB' \rightarrow$ as C.

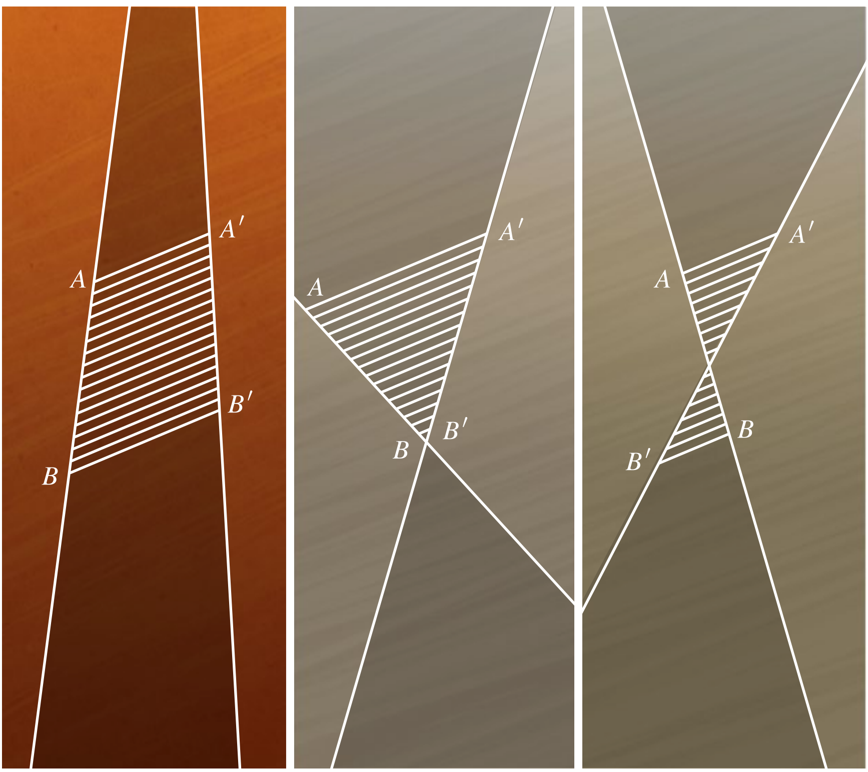


That means A' and C' have to be on opposite sides of $\leftarrow BB' \rightarrow$, and so the intersection of $\leftarrow BB' \rightarrow$ and $A'C'$, which is B', must be between A' and C'. $\square$

That's the story of how parallel projection and order interact. What about congruence?

THM: PARALLEL PROJECTION AND CONGRUENCE
Let $\Phi : \ell \to \ell'$ be a parallel projection. If a, b, A and B are all points on ℓ and if $ab \simeq AB$, then $a'b' \simeq A'B'$.

Proof. There are actually several scenarios here, depending upon the positions of the segments ab and AB relative to ℓ'. They could lie on the same side of ℓ', or they could lie on opposite sides of ℓ', or one or both could straddle ℓ', or one or both could have an endpoint on ℓ'. You have



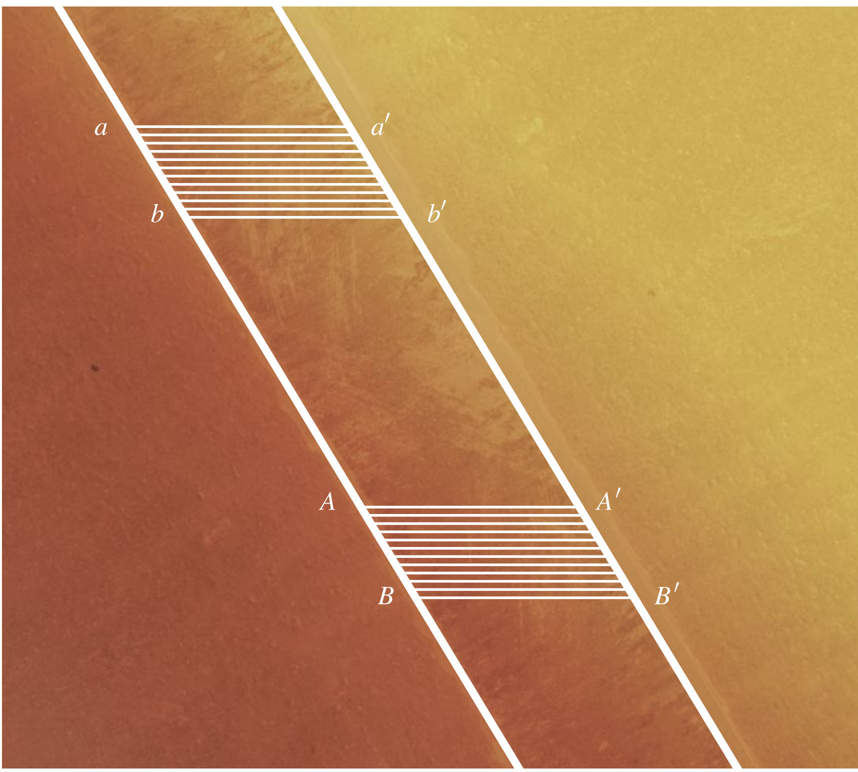
There are three positions for A and B relative to the image line– both on the same side, one on the image line, or one on each side. Likewise, there are three positions for a and b. Therefore, in all, there are nine scenarios.

to handle each of those scenarios slightly differently, but I am only going to address what I feel is the most iconic situation– the one where both segments are on the same side of ℓ'.

Case 1: ℓ and ℓ' are parallel.
First let's warm up with a simple case which I think helps illuminate the more general case– it is the case where ℓ and ℓ' are themselves parallel. Notice all the parallel line segments:

aa' is parallel to bb' and ab is parallel to $a'b'$ so $\square aa'b'b$ is a parallelogram;

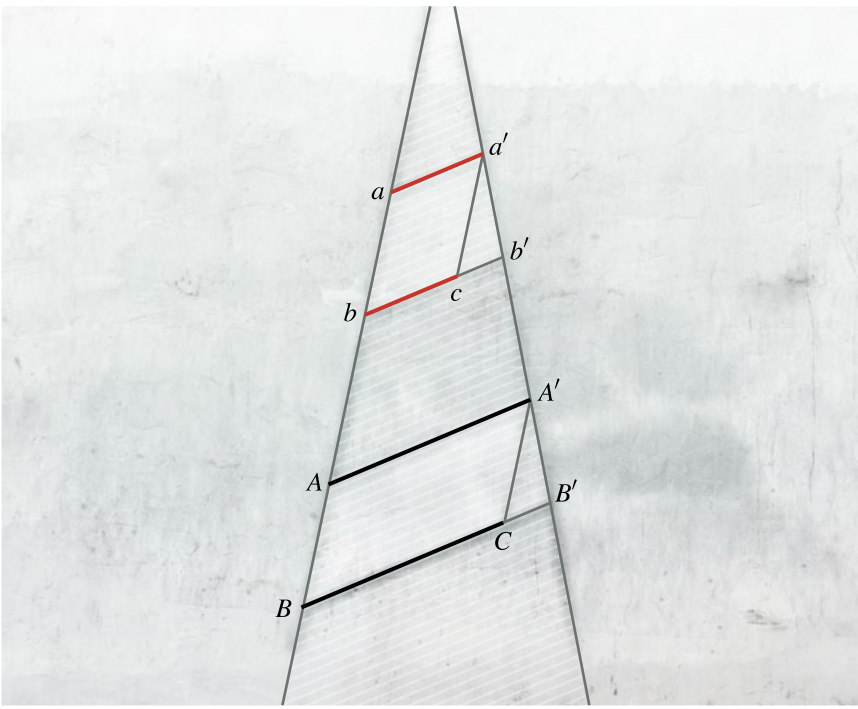
AA' is parallel to BB' and AB is parallel to $A'B'$ so $\square AA'B'B$ is also a parallelogram.



Case 1: when the two lines are parallel.

Because the opposite sides of a parallelogram are congruent (exercise 1 at the start of the lesson), $a'b' \simeq ab$ and $AB \simeq A'B'$. Since $ab \simeq AB$, that means $a'b' \simeq A'B'$.

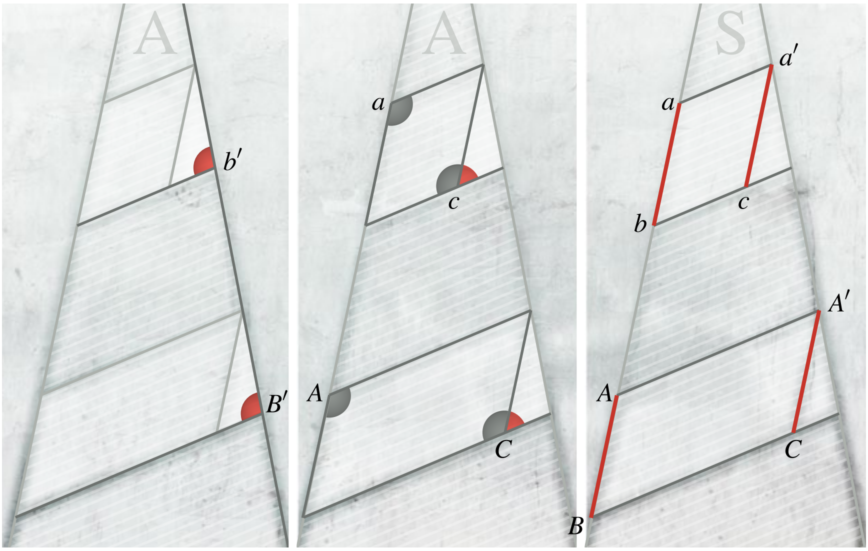
Case 2: ℓ and ℓ' are not parallel.
This is the far more likely scenario. In this case the two quadrilaterals $\square aa'b'b$ and $\square AA'B'B$ will not be parallelograms. I want to use the same approach here as in Case 1 though, so to do that we will need to build some parallelograms into the problem. Because ℓ and ℓ' are not parallel, the segments aa' and bb' cannot be the same length (exercise 3 at the start of this lesson), and the segments AA' and BB' cannot be the same length. Let's assume that aa' is shorter than bb' and that AA' is shorter than BB'. If this is not the case, then it is just a matter of switching some labels to make it so.



Then

- there is a point c between b and b' so that $bc \simeq aa'$, and
- there is a point C between B and B' so that $BC \simeq AA'$.

This creates four shapes of interest– the two quadrilaterals $\square a'abc$ and $\square A'ABC$ which are actually parallelograms (exercise 2), and the two triangles $\triangle a'b'c$ and $\triangle A'B'C$. The key here is to prove that $\triangle a'b'c \simeq \triangle A'B'C$. I want to use A·A·S to do that.



[A] $\angle b' \simeq \angle B'$.
The lines cb' and CB' are parallel (they are two of the projecting lines) and they are crossed by the tranversal ℓ'. By the converse of the Alternate Interior Angle Theorem, that means $\angle a'b'c$ and $\angle A'B'C$ are congruent.

[A] $\angle c \simeq \angle C$.
The opposite angles of the two parallelograms are congruent. Therefore $\angle a'cb \simeq \angle a'ab$ and $\angle A'AB \simeq \angle A'CB$. But aa' and AA' are parallel lines cut by the transversal ℓ, so $\angle a'ab \simeq \angle A'AB$. That means that $\angle a'cb \simeq \angle A'CB$, and so their supplements $\angle a'cb'$ and $\angle A'CB'$ are also congruent.

[S] $a'c \simeq A'C$.
The opposite sides of the two parallelograms are congruent too. Therefore $a'c \simeq ab$ and $AB \simeq A'C$, and since $ab \simeq AB$, that means $a'c \simeq A'C$.

By A·A·S, then, $\triangle a'b'c \simeq \triangle A'B'C$. The corresponding sides $a'b'$ and $A'B'$ have to be congruent.

□

Parallel projection and distance

That brings us to the question at the very heart of parallel projection. If Φ is a parallel projection and A and B are two points on ℓ, how do the lengths $|AB|$ and $|A'B'|$ compare? In Case 1 of the last proof, the segments AB and $A'B'$ ended up being congruent, but that was because ℓ and ℓ' were parallel. In general, AB and $A'B'$ do not have to be congruent. But (and this is the key) in the process of parallel projecting from one line to another, all distances are scaled by a constant multiple.

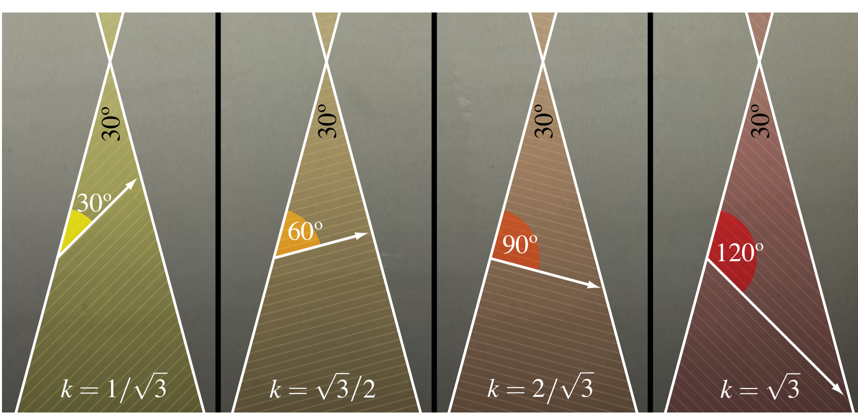
THM: PARALLEL PROJECTION AND DISTANCE

If $\Phi : \ell \to \ell'$ is a parallel projection, then there is a constant k such that

$$|A'B'| = k|AB|$$

for all points A and B on ℓ.

I want to talk about a few things before diving in after the formal proof. The first is that the previous theorem on congruence gives us a way to narrow the scope of the problem. Fix a point O on ℓ and let r be one of the two rays along ℓ with O as its endpoint. The Segment Construction Axiom says that every segment AB on ℓ is congruent to a segment OP where P is some point on r. We have just seen that parallel projection maps congruent segments to congruent segments. So if Φ scales all segments of the form OP by a factor of k, then it must scale all the segments of ℓ by that same factor.



Some parallel projections and their scaling constants.

The second deals with parallel projecting end-to-end congruent copies of a segment. For this, let me introduce another convenient notation convention: for the rest of this argument, when I write a point with a subscript P_d, the subscript d is the distance from that point to O. Now, pick a particular positive real value x, and let

$$k = |O'P'_x|/|OP_x|,$$

so that Φ scales the segment OP_x by a factor of k. Of course, eventually we will have to show that Φ scales all segments by that same factor, but for now let's restrict our attention to the segments OP_{nx}, where n is a positive integer. Between O and P_{nx} are $P_x, P_{2x}, \ldots P_{(n-1)x}$ in order:

$$O * P_x * P_{2x} * \cdots * P_{(n-1)x} * P_{nx}.$$

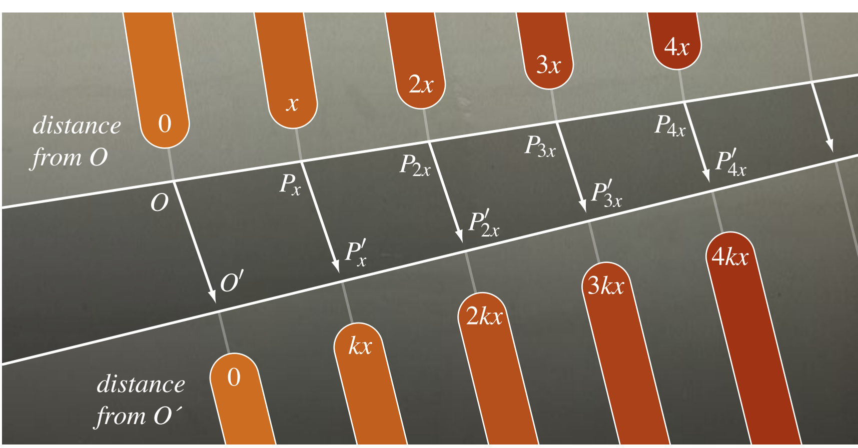
We have seen that parallel projection preserves the order of points, so

$$O' * P'_x * P'_{2x} * \cdots * P'_{(n-1)x} * P'_{nx}.$$

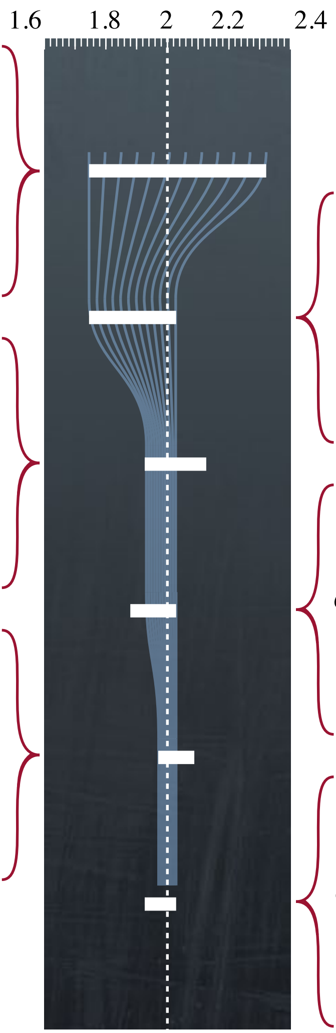
Each segment $P_{ix}P_{(i+1)x}$ is congruent to OP_x and consequently each parallel projection $P'_{ix}P'_{(i+1)x}$ is congruent to $O'P'_x$. Just add them all together

$$\begin{aligned}
|O'P'_{nx}| &= |O'P'_x| + |P'_xP'_{2x}| + |P'_{2x}P'_{3x}| + \cdots + |P'_{(n-1)x}P'_{nx}| \\
&= kx + kx + kx + \cdots + kx \quad (n \text{ times}) \\
&= k \cdot nx
\end{aligned}$$

and so Φ scales OP_{nx} by a factor of k.



Sadly, no matter what x is, the points P_{nx} account for an essentially inconsequential portion of the set of all points of r. However, if OP_x and OP_y were to have two different scaling factors we could use this end-to-end copying to magnify the difference between them. The third thing I would like to do, then, is to look at an example to see how this actually works, and how this ultmately prevents there from being two different scaling factors. In this example, let's suppose that $|O'P_1'| = 2$, so that all integer length segments on ℓ are scaled by a factor of 2, and let's take a look at what this means for $P_{3.45}$. Let k be the scaling factor for $OP_{3.45}$ and let's see what the first few end-to-end copies of $OP_{3.45}$ tell us about k.



1 $3 < 3.45 < 4$
$O * P_3 * P_{3.45} * P_4$
$O' * P_3' * P_{3.45}' * P_4'$
$6 < 3.45k < 8$
$1.74 < k < 2.32$

$6 < 6.9 < 7$ 2
$O * P_6 * P_{6.9} * P_7$
$O' * P_6' * P_{6.9}' * P_7'$
$12 < 6.9k < 14$
$1.74 < k < 2.03$

3 $10 < 10.35 < 11$
$O * P_{10} * P_{10.35} * P_{11}$
$O' * P_{10}' * P_{10.35}' * P_{11}'$
$20 < 10.35k < 22$
$1.93 < k < 2.13$

$13 < 13.8 < 14$ 4
$O * P_{13} * P_{13.8} * P_{14}$
$O' * P_{13}' * P_{13.8}' * P_{14}'$
$26 < 13.8k < 28$
$1.88 < k < 2.03$

5 $17 < 17.25 < 18$
$O * P_{17} * P_{17.25} * P_{18}$
$O' * P_{17}' * P_{17.25}' * P_{18}'$
$34 < 17.25k < 36$
$1.97 < k < 2.09$

$20 < 20.7 < 21$ 6
$O * P_{20} * P_{20.7} * P_{21}$
$O' * P_{20}' * P_{20.7}' * P_{21}'$
$40 < 20.7k < 42$
$1.93 < k < 2.03$

notation

The floor function, $f(x) = \lfloor x \rfloor$, assigns to each real number x the largest integer which is less than or equal to it.

The ceiling function, $f(x) = \lceil x \rceil$, assigns to each real number x the smallest integer which is greater than or equal to it.

Proof. It is finally time to prove that parallel projection scales distance. Let $k = |O'P_1'|$ so that k is the scaling factor for the segment of length one (and consequently all integer length segments). Now take some arbitrary point P_x on ℓ and let k' be the scaling factor for the segment OP_x. We want to show that $k' = k$ and to do that, I want to follow the same basic strategy as in the example above– capture k' in an increasingly narrow band around k by looking at the parallel projection of P_{nx} as n increases.

$$\lfloor nx \rfloor < nx < \lceil nx \rceil$$
$$O * P_{\lfloor nx \rfloor} * P_{nx} * P_{\lceil nx \rceil}$$
$$O' * P'_{\lfloor nx \rfloor} * P'_{nx} * P'_{\lceil nx \rceil}$$
$$k\lfloor nx \rfloor < k'nx < k\lceil nx \rceil$$

$$k(nx-1) < k\lfloor nx \rfloor < k'nx < k\lceil nx \rceil < k(nx+1)$$ *

$$k(nx-1) < k'nx < k(nx+1)$$
$$k \cdot (nx-1)/(nx) < k' < k \cdot (nx+1)/(nx)$$

As n increases, the two ratios $(nx-1)/(nx)$ and $(nx+1)/(nx)$ both approach 1. In the limit as n goes to infinity, they are one. Since the above inequalities have to be true for all n, the only possible value for k' , then, is k. □

* *In this step, I have replaced one set of inequalities with another, less precise, set. The new inequalities are easier to manipulate mathematically though, and are still accurate enough to get the desired result.*

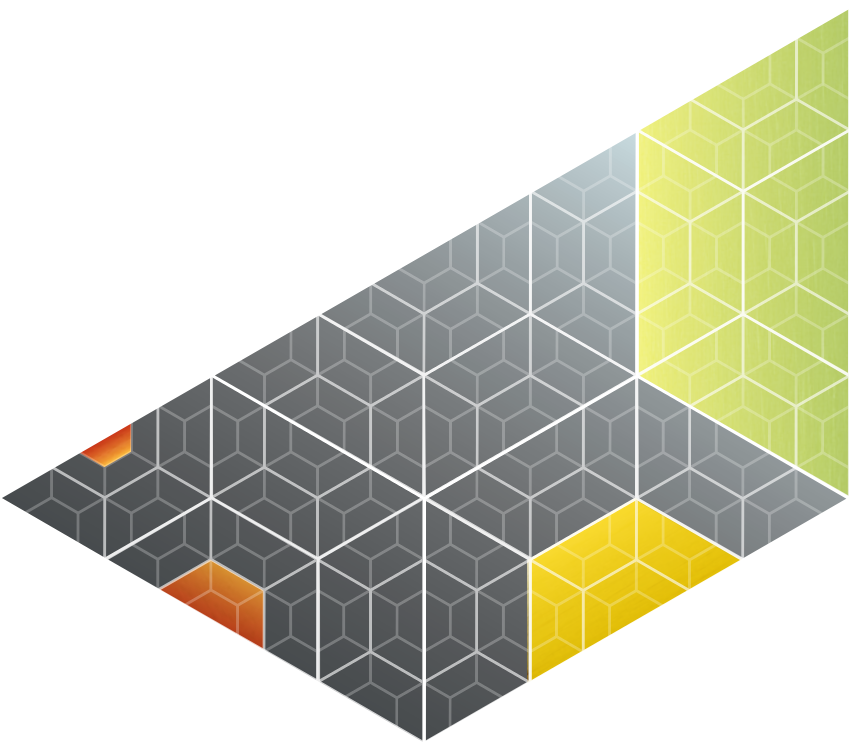
Exercises

1. Investigate the other possible cases in the proof that parallel projection preserves order.

2. Suppose that Φ is a parallel projection from ℓ to ℓ'. If ℓ and ℓ' intersect, and that point of intersection is P, prove that $\Phi(P) = P$.

3. Prove that if ℓ and ℓ' are parallel, then the scaling factor of any parallel projection between them must be one, but that if ℓ and ℓ' are not parallel, then there is a parallel projection with every possible scaling factor k where $0 < k < \infty$.

4. In the lesson 7, we constructed a distance function, and one of the keys to that construction was locating the points on a ray which were a distance of $m/2^n$ from its endpoint. In Euclidean geometry, there is a construction which locates all the points on a ray which are *any* rational distance m/n from its endpoint. Take two (non-opposite) rays r and r' with a common endpoint O. Along r, lay out m congruent copies of a segment of length one, ending at the point P_m. Along r', lay out n congruent copies of a segment of length one, ending at the point Q_n. Mark the point Q_1 on r' which is a distance one from O. Verify that the line which passes through Q_1 and is parallel to P_mQ_n intersects r a distance of m/n from O.



In the lessons on neutral geometry, we spent a lot of effort to gain an understanding of polygon congruence. In particular, I think we were pretty thorough in our investigation of triangle and quadrilateral congruence. So I sincerely hope that you haven't forgotten what it means for two polygons to be congruent:

1. all their corresponding sides must be congruent, and
2. all their corresponding interior angles must be congruent.

Remember as well that polygon congruence is an equivalence relation (it is reflexive, symmetric, and transitive). It turns out that congruence is not the only important equivalence relation between polygons, though, and the purpose of this lesson is to investigate another: similarity.

Similarity is a less demanding relation than congruence. I think of congruent polygons as exactly the same, just positioned differently. I think of similar polygons as "scaled versions" of one another– the same shape, but possibly different sizes. That's not really a definition, though, so let's get to something a little more formal.

DEF: SIMILAR POLYGONS

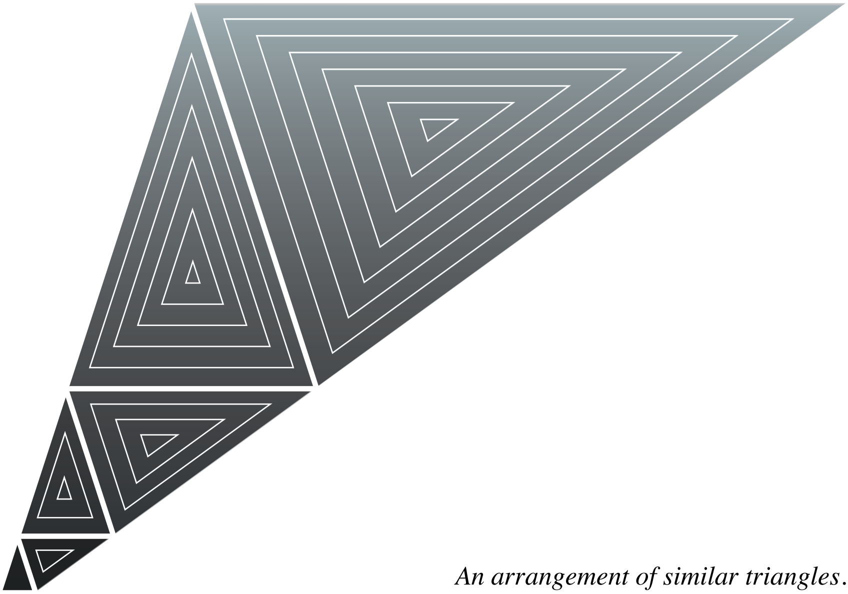
Two n-sided polygons $P_1P_2 \dots P_n$ and $Q_1Q_2 \dots Q_n$ are *similar* to one another if they meet two sets of conditions

1. corresponding interior angles are congruent:

$$\angle P_i \simeq \angle Q_i, \quad 1 \le i \le n.$$

2. corresponding side lengths differ by the same constant multiple:

$$|P_iP_{i+1}| = k \cdot |Q_iQ_{i+1}|, \quad 1 \le i \le n.$$



An arrangement of similar triangles.

I will use the notation $P_1P_2 \ldots P_n \sim Q_1Q_2 \ldots Q_n$ to indicate similarity. There are a few things worth noting here. First, if polygons are congruent, they will be similar as well– the scaling constant k will be one in this case. Second, similarity *is* an equivalence relation– I leave it to you to verify that the three required conditions are met. Third, when you jump from one polygon to another similar polygon, all the corresponding segments lengths are scaled by the same amount. That behavior echoes the work we did in the last lesson, and for good reason: parallel projection underlies everything that we are going to do in this lesson.

A spiralling stack of similar golden rectangles (see the exercises).

Much of the time, when working with either parallel projection or similarity, the actual scaling constant is just not that important. The only thing that matters is that there *is* a scaling constant. Fortunately, the existence of a scaling constant can be indicated without ever mentioning what it is. The key to doing this is ratios. Consider a parallel projection from line ℓ to line ℓ'. Let A, B, a, and b be points on ℓ and let A', B', a' and b' be their respective images on ℓ'. The main result of the last lesson was that there is a scaling constant k so that

$$|A'B'| = k \cdot |AB| \quad \& \quad |a'b'| = k \cdot |ab|.$$

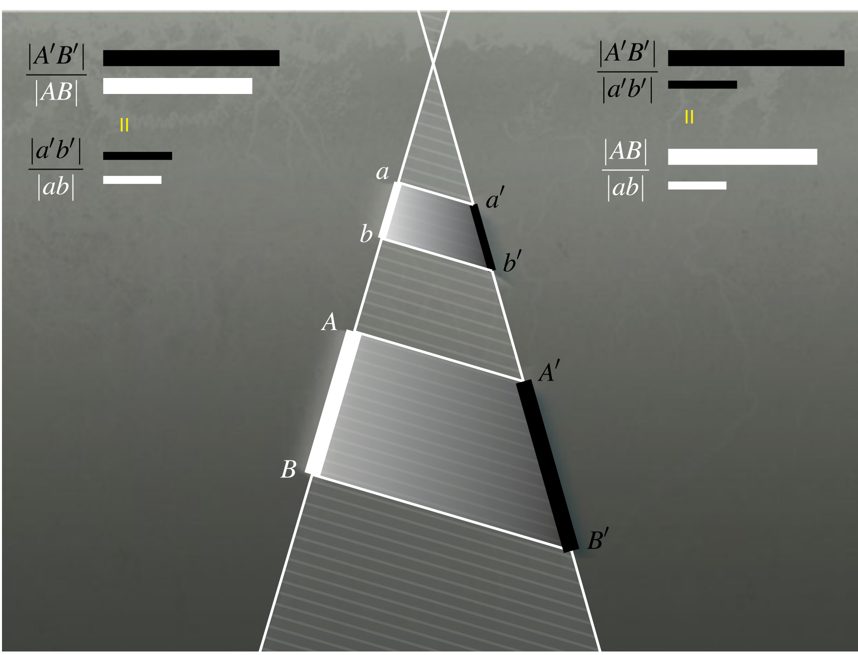
The ratios I am talking about are only a step away from this pair of equations.

Ratio 1: Solve for k in both equations and set them equal to each other

$$\frac{|A'B'|}{|AB|} = \frac{|a'b'|}{|ab|}.$$

Ratio 2: Starting from the first ratio, multiply through by $|AB|$ and divide through by $|a'b'|$

$$\frac{|A'B'|}{|a'b'|} = \frac{|AB|}{|ab|}.$$



Two invariant ratios of a parallel projection.

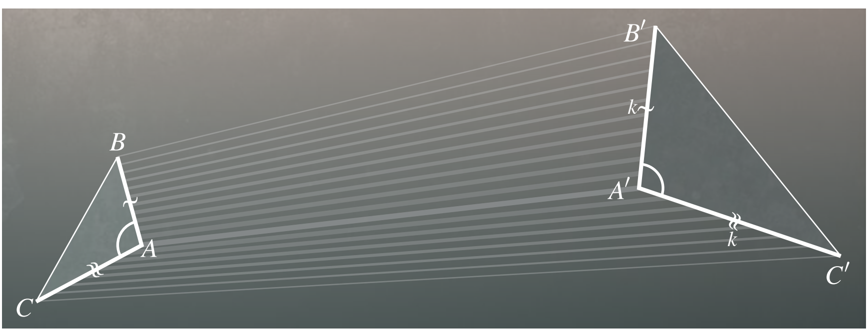
Triangle similarity theorems

I would now like to turn our attention to a few theorems that deal with similarity of triangles. I like to think of these similarity theorems as degenerations of the triangle congruence theorems, where the strict condition of side congruence, $A'B' \simeq AB$, is replaced with the more flexible condition of constant scaling, $|A'B'| = k|AB|$. First up is the S·A·S similarity theorem.

THM: S·A·S SIMILARITY
In triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\angle A \simeq \angle A'$ and if there is a constant k so that

$$|A'B'| = k \cdot |AB| \quad \& \quad |A'C'| = k \cdot |AC|,$$

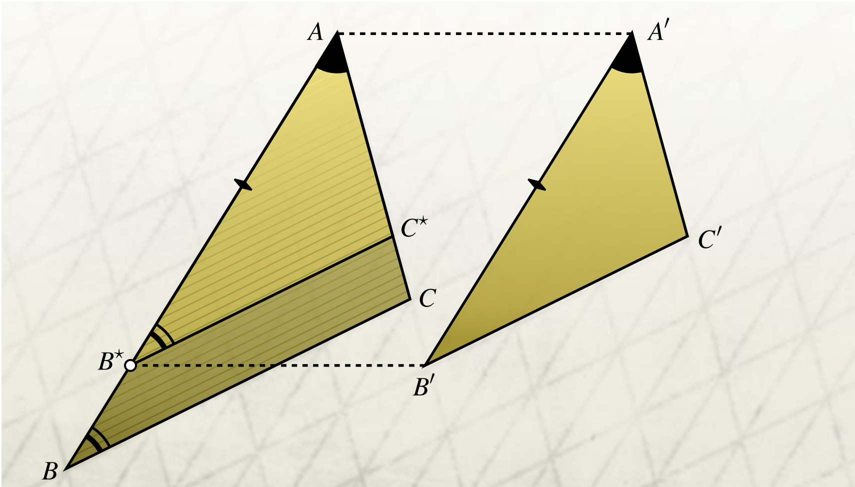
then $\triangle ABC \sim \triangle A'B'C'$.



Proof. First of all, let me point out that just as with the parallel projection, the second condition in the S·A·S similarity theorem can be recast in terms of ratios:

$$\begin{cases} |A'B'| = k|AB| \\ |A'C'| = k|AC| \end{cases} \iff \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} \iff \frac{|A'B'|}{|A'C'|} = \frac{|AB|}{|AC|}.$$

With that said, what we need to do in this proof is to establish two more angle congruences, that $\angle B \simeq \angle B'$ and $\angle C \simeq \angle C'$, and one more ratio of sides, that $|B'C'| = k|BC|$. Two parallel projections will form the backbone of this proof. The first will establish the two angle congrunces while the second will calculate the ratio of the third pair of sides.



The first parallel projection. The primary purpose of the first projection is to build a transitional triangle which is congruent to $\triangle A'B'C'$ but positioned on top of $\triangle ABC$. Begin by locating the point $B^\star$ on $AB\rightarrow$ so that $AB^\star \simeq A'B'$. We cannot know the exact location of $B^\star$ relative to B on this ray– that depends upon whether $A'B'$ is shorter or longer than AB. For this argument, assume that $A'B'$ is shorter than AB, which will place $B^\star$ between A and B (the other case is not substantially different). Consider the parallel projection

$$\Phi_1 : (\leftarrow AB \rightarrow) \longrightarrow (\leftarrow AC \rightarrow)$$

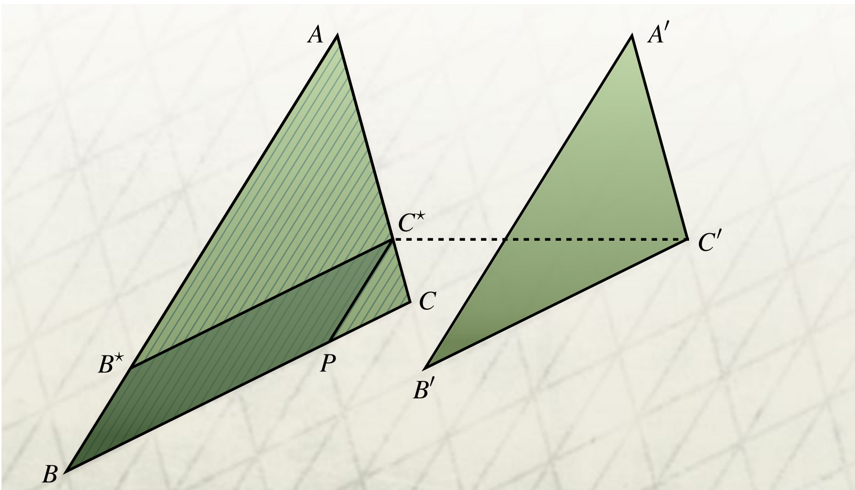
which takes B to C. Note that since A is the intersection of these two lines, $\Phi_1(A) = A$. Label $C^\star = \Phi_1(B^\star)$. Let's see how the newly formed $\triangle AB^\star C^\star$ compares with $\triangle A'B'C'$. Compare the ratios

$$\frac{|AC^\star|}{|AC|} \overset{1}{=} \frac{|AB^\star|}{|AB|} \overset{2}{=} \frac{|A'B'|}{|AB|} \overset{3}{=} \frac{|A'C'|}{|AC|}.$$

1. parallel projection
2. constructed congruence
3. given

If you look at the first and last entries in that string of equalities you will see that $|AC^\star| = |A'C'|$. Put that together with what we already knew, that $AB^\star \simeq A'B'$ and $\angle A \simeq \angle A'$, and by S·A·S, we see that $\triangle A'B'C'$ and $\triangle AB^\star C^\star$ are congruent. In particular, that means $\angle B' \simeq \angle B^\star$ and $\angle C' \simeq$

$\angle C^\star$. Now let's turn back to see how $\triangle AB^\star C^\star$ relates to $\triangle ABC$. In order to locate $C^\star$, we used a projection which was parallel to $\leftarrow BC \rightarrow$. That of course means $\leftarrow B^\star C^\star \rightarrow$ and $\leftarrow BC \rightarrow$ are parallel to one another, and so, by the converse of the Alternate Interior Angle Theorem, $\angle B^\star \simeq \angle B$ and $\angle C^\star \simeq \angle C$. Since angle congruence is transitive, we now have the two required angle congruences, $\angle B \simeq \angle B'$ and $\angle C \simeq \angle C'$.



The second parallel projection. Consider the parallel projection

$$\Phi_2 : (\leftarrow AC \rightarrow) \longrightarrow (\leftarrow BC \rightarrow)$$

which maps A to B. Again, since C is the intersection of those two lines, $\Phi_2(C) = C$. The other point of interest this time is $C^\star$. Define $P = \Phi_2(C^\star)$. In doing so, we have effectively carved out a parallelogram $BB^\star C^\star P$. Recall that the opposite sides of a parallelogram are congruent– in particular, $B^\star C^\star \simeq BP$. Now consider the ratios that Φ_2 provides

$$\frac{|B'C'|}{|BC|} \overset{1}{=} \frac{|B^\star C^\star|}{|BC|} \overset{2}{=} \frac{|BP|}{|BC|} \overset{3}{=} \frac{|AC^\star|}{|AC|} \overset{4}{=} \frac{|A'C'|}{|AC|} = k.$$

1. triangle congruence established above
2. opposite sides of a parallelogram
3. parallel projection
4. triangle congruence established above

Thus, $|B'C'| = k|BC|$, as needed. $\square$

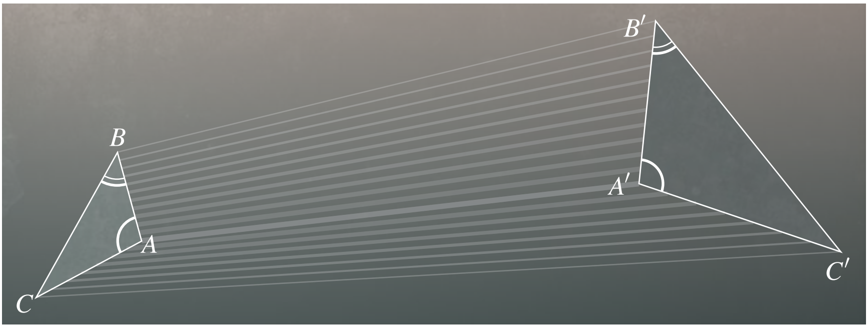
Back in the neutral geometry lessons, after S·A·S we next encountered A·S·A and A·A·S. Unlike S·A·S, both of those theorems reference only one pair of sides in the triangles. Let's take a look at what happens when you try to modify those congruence conditions into similarity conditions.

A·S·A Congruence	A·S·A Similarity (?)
$\angle A \simeq \angle A'$	$\angle A \simeq \angle A'$
$AB \simeq A'B'$	$\lvert A'B' \rvert = k \cdot \lvert AB \rvert$
$\angle B \simeq \angle B'$	$\angle B \simeq \angle B'$

A·A·S Congruence	A·A·S Similarity (?)
$\angle A \simeq \angle A'$	$\angle A \simeq \angle A'$
$\angle B \simeq \angle B'$	$\angle B \simeq \angle B'$
$BC \simeq B'C'$	$\lvert B'C' \rvert = k \cdot \lvert BC \rvert$

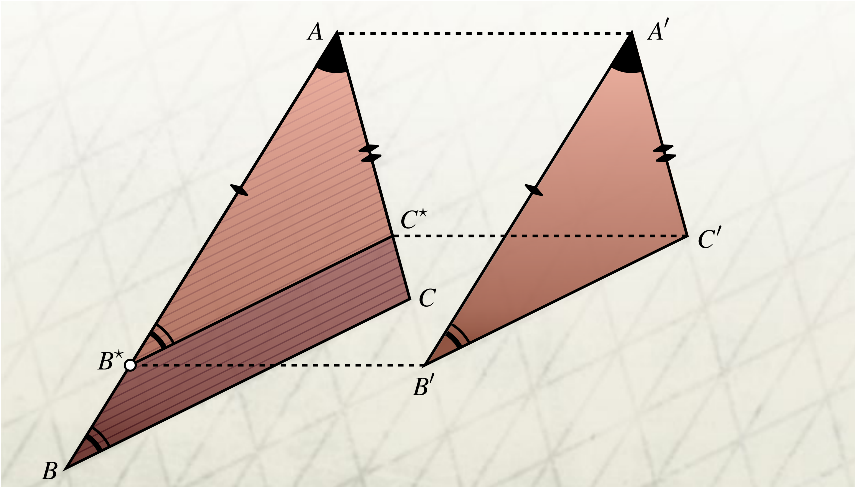
In each of these conversions, the condition on the one side is automatically satisfied– there will always be a real value of k that makes the equation true. That is a hint that it may take only two angle congruences to guarantee similarity.

THM: A · A SIMILARITY
In triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\angle A \simeq \angle A'$ and $\angle B \simeq \angle B'$, then $\triangle ABC \sim \triangle A'B'C'$.



Proof. We have plenty of information about the angles, so what we need here is some information about ratios of sides. In particular, I want to show that

$$\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|}.$$



Along with the given congruence $\angle A \simeq \angle A'$, that will be enough to use S·A·S similarity. As in the S·A·S similarity proof, I want to construct a transition triangle: one that is positioned on top of $\triangle ABC$ but is congruent to $\triangle A'B'C'$. To do that, locate $B^\star$ on $AB\rightarrow$ so that $AB^\star \simeq A'B'$, and $C^\star$ on $AC\rightarrow$ so that $AC^\star \simeq A'C'$. By S·A·S, $\triangle AB^\star C^\star$ and $\triangle A'B'C'$ are congruent. Now take a look at all the congruent angles

$$\angle B^\star \simeq \angle B' \simeq \angle B.$$

According to the Alternate Interior Angle Theorem, $\leftarrow B^\star C^\star \rightarrow$ and $\leftarrow BC \rightarrow$ must be parallel. Therefore the parallel projection from $\leftarrow AB \rightarrow$ to $\leftarrow AC \rightarrow$ which maps B to C and A to itself will also map $B^\star$ to $C^\star$. That gives us some ratios

$$\frac{|A'B'|}{|AB|} \stackrel{1}{=} \frac{|AB^\star|}{|AB|} \stackrel{2}{=} \frac{|AC^\star|}{|AC|} \stackrel{3}{=} \frac{|A'C'|}{|AC|}.$$

1. constructed congruence
2. parallel projection
3. constructed congruence

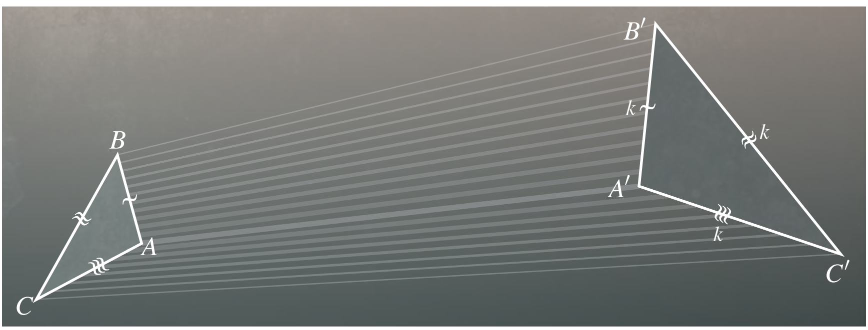
The first and last terms in that list of equalities give the ratio we need. That, together with the known congruence $\angle A \simeq \angle A'$, is enough for S·A·S similarity, so $\triangle ABC \sim \triangle A'B'C'$. $\square$

Note that while A·A·A was not enough to guarantee congruence, thanks to the result above, we now know that it is (more than) enough to guarantee similarity. Finally, the last of the triangle similarity theorems is S·S·S similarity (S·S·A, which just misses as a congruence theorem, is done in again by the same counterexample).

THM: S·S·S SIMILARITY
In triangles $\triangle ABC$ and $\triangle A'B'C'$, if there is a constant k so that

$$|A'B'| = k \cdot |AB| \quad |B'C'| = k \cdot |BC| \quad \& \quad |C'A'| = k \cdot |CA|,$$

then $\triangle ABC \sim \triangle A'B'C'$.



I am going to leave the proof of this last similarity theorem as an exercise for you.

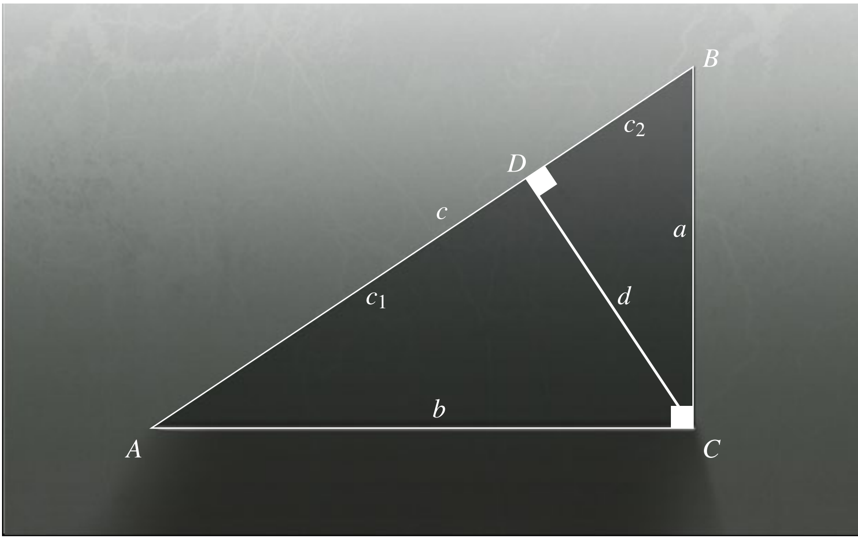
The Pythagorean Theorem

Before we close this lesson, though, let's meet one of the real celebrities of the subject.

THM: THE PYTHAGOREAN THEOREM
Let $\triangle ABC$ be a right triangle whose right angle is at the vertex C. Identify the lengths of each side as

$$a = |BC| \quad b = |AC| \quad c = |AB|.$$

Then $c^2 = a^2 + b^2$.



A proof of the Pythagorean Theorem via similarity.

Proof. There are many, many proofs of this theorem. The one that I am going to give involves dividing the triangle into two smaller triangles, showing each of those is similar to the initial triangle, and then working with ratios. Let D be the foot of the perpendicular to AB through C. The segment CD divides $\triangle ABC$ into two smaller triangles: $\triangle ACD$ and $\triangle BCD$. Let's go ahead and label the lengths of the newly created sides of those two triangles:

$$c_1 = |AD| \quad c_2 = |BD| \quad d = |CD|$$

and note that $c = c_1 + c_2$. Now $\triangle ADC$ shares $\angle A$ with $\triangle ACB$, and they both have a right angle, so by the A·A similarity theorem, $\triangle ADC \sim \triangle ACB$. Similarly, $\triangle BDC$ shares $\angle B$ with $\triangle ACB$, and they both have a right angle as well, so again by A·A similarity, $\triangle BDC \sim \triangle ACB$. From these similarities, there are many ratios, but the two that we need are

$$\frac{a}{c} = \frac{c_2}{a} \implies a^2 = c \cdot c_2 \quad \& \quad \frac{b}{c} = \frac{c_1}{b} \implies b^2 = c \cdot c_1.$$

Now all you have to do is add those two equations together and simplify to get the Pythagorean Theorem

$$a^2 + b^2 = c \cdot c_2 + c \cdot c_1 = c(c_2 + c_1) = c^2.$$

$\square$

Exercises

1. Prove that similarity of polygons is an equivalence relation.

2. Prove the S·S·S triangle similarity theorem.

3. State and prove the S·A·S·A·S and A·S·A·S·A similarity theorems for convex quadrilaterals.

The six trigonometric functions can be defined, for values of θ between 0 and 90°, as ratios of pairs of sides of a right triangle with an interior angle θ. If the length of the hypotenuse is h, the length of the leg adjacent to θ is a, and the length of the leg opposite θ is o, then these functions are defined as

$$\begin{aligned}
&\text{sine: } \sin(\theta) = o/h \\
&\text{cosine: } \cos(\theta) = a/h \\
&\text{tangent: } \tan(\theta) = o/a \\
&\text{cotangent: } \cot(\theta) = a/o \\
&\text{secant: } \sec(\theta) = h/a \\
&\text{cosecant: } \csc(\theta) = h/o.
\end{aligned}$$

4. Verify that the six trigonometric functions are well-defined. That is, show that it does not matter which right triangle with interior angle θ you choose– these six ratios will not change.

5. Verify the Pythagorean identities (for values of θ between 0 and 90°).

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
1 + \tan^2\theta &= \sec^2\theta \\
1 + \cot^2\theta &= \csc^2\theta
\end{aligned}$$

6. Verify the cofunction identities (for values of θ between 0 and 90°).

$$\begin{aligned}
\sin(90^\circ - \theta) &= \cos\theta \\
\cos(90^\circ - \theta) &= \sin\theta \\
\tan(90^\circ - \theta) &= \cot\theta \\
\cot(90^\circ - \theta) &= \tan\theta \\
\sec(90^\circ - \theta) &= \csc\theta \\
\csc(90^\circ - \theta) &= \sec\theta
\end{aligned}$$

7. The geometric mean of two numbers a and b is defined to be $\sqrt{ab}$. Let $\triangle ABC$ be a right triangle with right angle at C and let D be the point on AB so that CD is perpendicular to AB (the same setup as in the proof of the Pythagorean Theorem). Verify that $|CD|$ is the geometric mean of $|AD|$ and $|BD|$.

8. Consider a rectangle $\square ABCD$ with $|AB| < |BC|$, and suppose that this rectangle has the following special property: if a square $\square ABEF$ is constructed inside $\square ABCD$, then the remaining rectangle $\square ECDF$ is similar to the original $\square ABCD$. A rectangle with this property is called a *golden rectangle*. Find the value of $|BC|/|AB|$, a value known as the golden ratio.

16 CIRCLES.

WHAT GOES
AROUND...

This is the first of two lessons dealing with circles. This lesson gives some basic definitions and some elementary theorems, the most important of which is the Inscribed Angle Theorem. In the next lesson, we will tackle the important issue of circumference and see how that leads to the radian angle measurement system.

Definitions

So you might be thinking "Lesson 16 and we are just now getting to circles... what was the hold-up?" In fact, we could have given a proper definition for the term *circle* as far back as lesson 3. All that you really need for a good definition is points, segments, and congruence. But once you give the definition, what next? Most of what I want to cover with circles is specific to Euclidean geometry. I don't know that many theorems about circles in neutral geometry, and in the discussion thus far, the only time I remember that the lack of circles made things awkward was when we looked at cyclic polygons. In any case, now *is* the time, so

> DEF: CIRCLE
> For any point O and positive real number r, the *circle* with *center* O and *radius* r is the set of points which are a distance r from O.

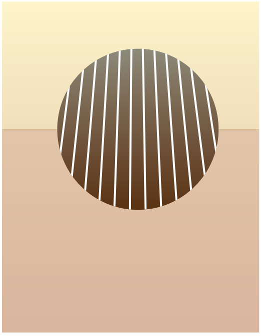
A few observations.

1. A circle is a set. Therefore, you should probably speak of the elements of that set as the points *of* the circle, but it is more common to refer to these as points *on* the circle.

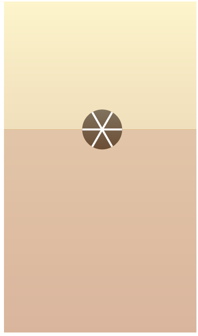
2. In the definition I have given, the radius is a number. We often talk about the radius as a geometric entity though– as one of the segments from the center to a point on the circle.

3. We tend to think of the center of a circle as a fundamental part of it, but you should notice that the center of a circle is not actually a point on the circle.

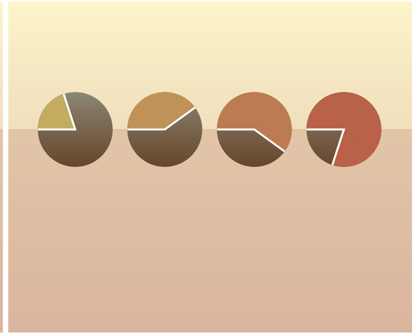
4. It is not that common to talk about circles as congruent or not congruent. If you were to do it, though, you would say that two circles are congruent if and only if they have the same radius.



12 chords



3 diameters



4 central angles

Before we get into anything really complicated, let's get a few other related definitions out of the way.

DEF: CHORD AND DIAMETER

A segment with both endpoints on a circle is called a *chord* of that circle. A chord which passes through the center of the circle is called a *diameter* of that circle.

Just like the term radius, the term diameter plays two roles, a numerical one and geometric one. The diameter in the numerical sense is just the length of the diameter in the geometric sense.

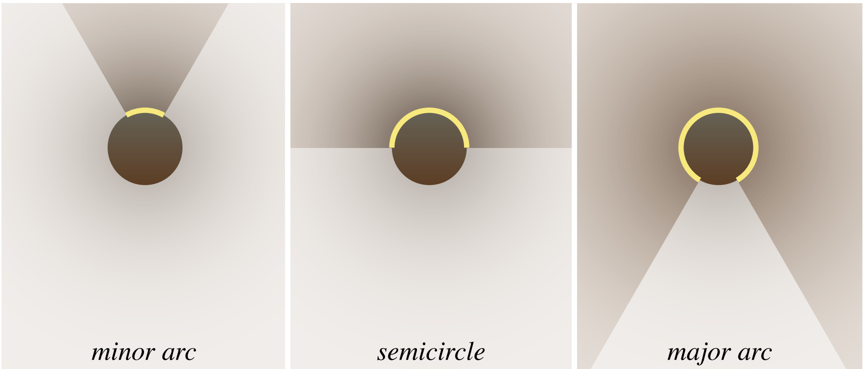
DEF: CENTRAL ANGLE

An angle with its vertex at the center of a circle is called a *central angle* of that circle.

We will see (in the next section) that a line intersects a circle at most twice. Therefore, if AB is a chord of a circle, then all the points of that circle other than A and B are on one side or the other of $\leftarrow AB \rightarrow$. Thus $\leftarrow AB \rightarrow$ separates those points into two sets. These sets are called *arcs* of the circle. There are three types of arcs– semicircles, major arcs, and minor arcs– depending upon where the chord crosses the circle.

DEF: SEMICIRCLE

Let AB be a diameter of a circle $\mathcal{C}$. All the points of $\mathcal{C}$ which are on one side of $\leftarrow AB \rightarrow$, together with the endpoints A and B, form a *semicircle*.



Each diameter divides the circle into two semicircles, overlapping at the endpoints A and B.

DEF: MAJOR AND MINOR ARC

Let AB be a chord of a circle $\mathcal{C}$ which is not a diameter, and let O be the center of this circle. All the points of $\mathcal{C}$ which are on the same side of $\leftarrow AB \rightarrow$ as O, together with the endpoints A and B, form a *major arc*. All the points of $\mathcal{C}$ which are on the opposite side of $\leftarrow AB \rightarrow$ from O, together with the endpoints A and B, form a *minor arc*.

Like the two semicircles defined by a diameter, the major and minor arcs defined by a chord overlap only at the endpoints A and B. For arcs in general, including diameters, I use the notation $\smile AB$. Most of the arcs we look at will be minor arcs, so in the instances when I want to emphasize that we are looking at a major arc, I will use the notation $\frown AB$.

There is a very simple, direct, and important relationship between arcs and central angles. You may recall that in the lesson on polygons, I suggested that two rays with a common endpoint define not one, but two angles– a "proper" angle and a "reflex" angle. These proper and reflex angles are related to the minor and major arcs as described in the next theorem, whose proof I leave to you.

THM: CENTRAL ANGLES AND ARCS

Let AB be a chord of a circle with center O. The points of $\smile AB$ are A, B, and all the points in the interior of the proper angle $\angle AOB$. The points of $\frown AB$ are A, B, and all the points in the interior of the reflex angle $\angle AOB$ (that is, the points exterior to the proper angle).

Intersections

Circles are different from the shapes we have been studying to this point because they are not built out of lines or line segments. Circles do share at least one characteristic with simple polygons though– they have an interior and an exterior. For any circle $\mathcal{C}$ with center O and radius r, and for any point P which is not $\mathcal{C}$,

- if $|OP| < r$, then P is inside $\mathcal{C}$;
- if $|OP| > r$, then P is outside $\mathcal{C}$.

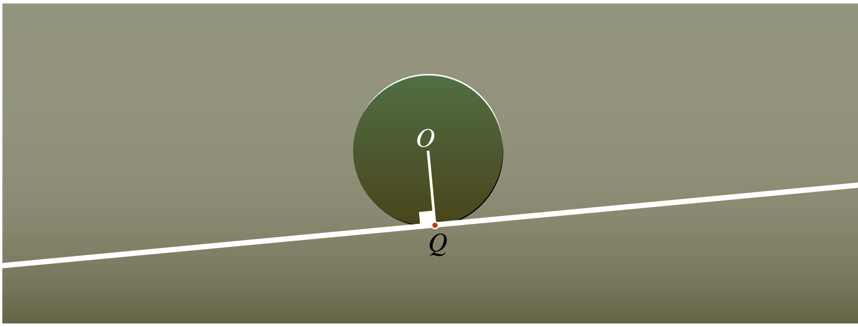
The set of points inside the circle is the interior and the set of points outside the circle is the exterior. Just like simple polygons, the circle separates the interior and exterior from each other. To get a better sense of that, we need to look at how circles intersect other basic geometric objects.

THM: A LINE AND A CIRCLE
A line will intersect a circle in 0, 1, or 2 points.

Proof. Let O be the center of a circle $\mathcal{C}$ of radius r, and let ℓ be a line. It is easy to find points on ℓ that are very far from $\mathcal{C}$, but are there any points on ℓ that are close to $\mathcal{C}$? The easiest way to figure out how close ℓ gets to $\mathcal{C}$ is to look at the closest point on ℓ to the center O. We saw (it was a lemma for the proof of A·A·A·S·S in lesson 10) that the closest point to O on ℓ is the foot of the perpendicular– call this point Q.

Zero intersections: $|OQ| > r$.
All the other points of ℓ are even farther from O, so none of the points on ℓ can be on $\mathcal{C}$.

One intersection: $|OQ| = r$.
Of course Q is an intersection, but it is the only intersection because all the other points on ℓ are farther away from O.



Two intersections: $|OQ| < r$.
The line spends time both inside and outside the circle. We just need to find where the line crosses in, and then back out of, the circle. The idea is to relate a point's distance from O to its distance from Q, and we can do that with the Pythagorean Theorem. If P is any point on ℓ other than Q, then $\triangle OQP$ will be a right triangle with side lengths that are related by the Pythagorean theorem

$$|OQ|^2 + |QP|^2 = |OP|^2.$$

In order for P to be on the circle, $|OP|$ must be exactly r. That means that $|PQ|$ must be exactly $\sqrt{r^2 - |OQ|^2}$. Since $|OQ| < r$, this expression is a positive real number, and so there are exactly two points on ℓ, one on each side of Q, that are this distance from Q. $\square$

A line that intersects a circle once (at the foot of the perpendicular) is called a *tangent* line to the circle. A line that intersects a circle twice is called a *secant* line of the circle. There is a important corollary that turns this last theorem about lines into a related theorem about segments.

COR: A SEGMENT AND A CIRCLE

If point P is inside a circle, and point Q is outside it, then the segment PQ intersects the circle.

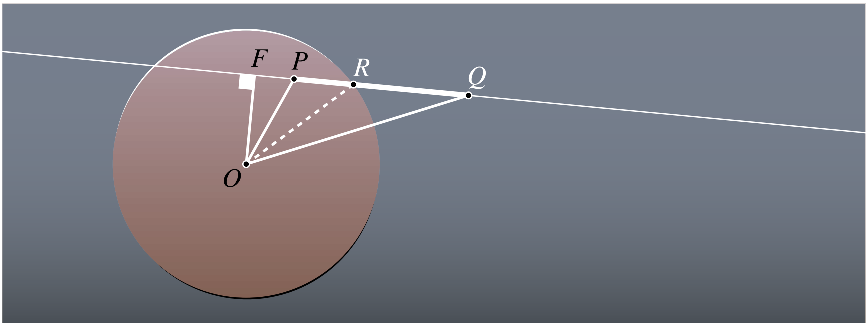
Proof. Label the center of the circle O. From the last theorem, we know that $\leftarrow PQ \rightarrow$ intersects the circle twice, and that the two intersections are separated by F, the foot of the perpendicular to PQ through O. The important intersection here is the one that is on the same side of the foot of the perpendicular as Q– call this point R. According to the Pythagorean theorem (with triangles $\triangle OFR$ and $\triangle OFQ$),

$$|FQ| = \sqrt{|OQ|^2 - |OF|^2} \quad \& \quad |FR| = \sqrt{|OR|^2 - |OF|^2}.$$

Since $|OQ| > |OR|$, $|FQ| > |FR|$, which places R between F and Q. We don't know whether P and Q are on the same side of F, though. If they are on opposite sides of F, then $P * F * R * Q$, so R is between P and Q as needed. If P and Q are on the same side of F, then we need to look at the right triangles $\triangle OFP$ and $\triangle OFR$. They tell us that

$$|FP| = \sqrt{|OP|^2 - |OF|^2} \quad \& \quad |FQ| = \sqrt{|OQ|^2 - |OF|^2}.$$

Since $|OP| < |OR|$, $|FP| < |FR|$, which places P between F and R. Finally, if P is between F and R, and R is between F and Q, then R has to be between P and Q. $\square$



There is another important question of intersections, and that involves the intersection of two circles. If two circles intersect, then it is highly likely their two centers and the point of intersection will be the vertices of a triangle (there is a chance the three could be colinear, and we will deal with that separately). The lengths of all three sides of that triangle will be known (the two radii and the distance between centers). So this question is not so much one about circles, but whether triangles can be built with three given side lengths. We have one very relevant result– the Triangle Inequality says that if a, b, and c are the lengths of the side of a triangle, then

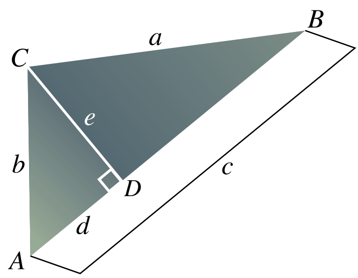
$$|a-b| < c < a+b.$$

What about the converse, though? If a, b, and c are any positive reals satisfying the Triangle Inequality conditions, can we put together a triangle with sides of those lengths? As much as a digression as it is, we need to answer this question before moving on.

THM: BUILDABLE TRIANGLES

Let a, b, and c be positive real numbers. Suppose that c is the largest of them and that $c < a+b$. Then there is a triangle with sides of length a, b, and c.

Proof. Start off with a segment AB whose length is c. We need to place a third point C so that it is a distance a from B and b from A. According to S·S·S, there is only one such triangle "up to congruence", so this may not be too easy. What I am going to do, though, is to build this triangle out of a couple of right triangles (so that I can use the Pythagorean theorem). Mark D on $AB\rightarrow$ and label $d = |AD|$. Mark C on one of the rays with endpoint D which is perpendicular to AB and label $e = |CD|$. Then both $\triangle ACD$ and $\triangle BCD$ are right triangles. Furthermore, by sliding D and C along their respective rays, we can make d and e any positive numbers.



We need to see if it is possible to position the two so that $|AC| = b$ and $|BC| = a$.

> To get $|AC| = b$, we will need $d^2 + e^2 = b^2$.
> To get $|BC| = a$, we will need $(c-d)^2 + e^2 = a^2$.

It's time for a little algebra to find d and e. According to the Pythagorean Theorem,

$$
\begin{gathered}
b^2 - d^2 = e^2 = a^2 - (c-d)^2 \\
b^2 - d^2 = a^2 - c^2 + 2cd - d^2 \\
b^2 = a^2 - c^2 + 2cd \\
(b^2 - a^2 + c^2)/2c = d.
\end{gathered}
$$

Since we initially required $c > a$, this will be a positive value. Now let's plug back in to find e.

$$
e^2 = b^2 - d^2 = b^2 - \left(\frac{b^2 - a^2 + c^2}{2c}\right)^2.
$$

Here is the essential part– because we will have to take a square root to find e, the right hand side of this equation has to be positive– otherwise the equation has no solution and the triangle cannot be built. Let's go back to see if the Triangle Inequality condition on the three sides will help:

$$
\begin{gathered}
c < a + b \\
c - b < a \\
(c-b)^2 < a^2 \\
c^2 - 2bc + b^2 < a^2 \\
c^2 - a^2 + b^2 < 2bc \\
(c^2 - a^2 + b^2)/2c < b \\
((c^2 - a^2 + b^2)/2c)^2 < b^2 \\
0 < b^2 - ((c^2 - a^2 + b^2)/2c)^2
\end{gathered}
$$

which is exactly what we want [of course, when I first did this calculation, I worked in the other direction, from the answer to the condition]. As long as $c < a + b$, then, a value for e can be found, and that means the triangle can be built. $\square$

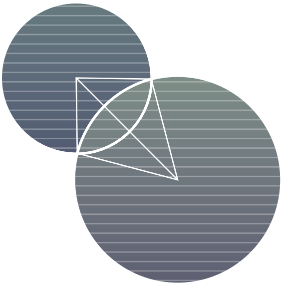
Now let's get back to the real issue at hand– that of the intersection of two circles.

THM: A CIRCLE AND A CIRCLE
Two circles intersect at 0, 1, or 2 points.

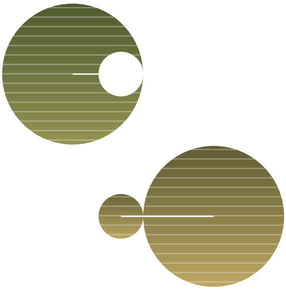
Proof. Three factors come in to play here: the radius of each circle and the distance between their centers. Label

r_1, r_2: the radii of the two circles, and
c, the distance between the centers.

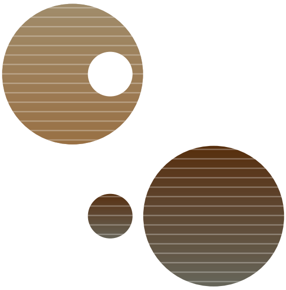
Two intersections:
when $|r_1 - r_2| < c < r_1 + r_2$.
There are exactly two triangles, $\triangle O_1XO_2$ and $\triangle O_1YO_2$, one on each side of O_1O_2, with sides of the required lengths. Therefore there are exactly two intersections of the two circles.



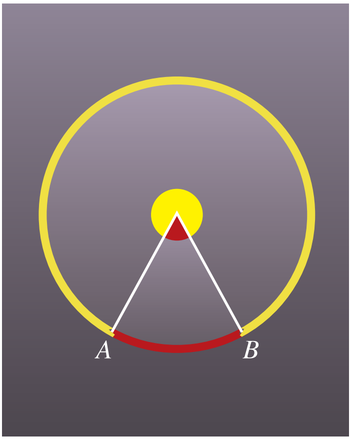
One intersection:
when $c = |r_1 - r_2|$ *or* $c = r_1 + r_2$.
In these two limiting cases, the triangle devolves into a line segment and the two intersections merge. In the first, either $O_1 * O_2 * X$ or $X * O_1 * O_2$, depending upon which radius is larger. In the second $O_1 * X * O_2$.



Zero intersections:
when $c < |r_1 - r_2|$ *or* $c > r_1 + r_2$.
In this case, you just cannot form the needed triangle (it would violate the Triangle Inequality), so there cannot be any intersections. In the first case, one circles lies entirely inside the other. In the second, they are separated from one another.



□



Major arc: reflex ∠AOB
Minor arc: proper ∠AOB

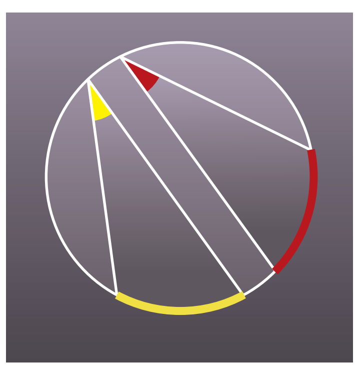
As I mentioned before, there is a one-to-one correspondence between central angles and arcs that matches the proper angle $\angle AOB$ with the minor arc $\smile AB$ and the reflex angle $\angle AOB$ with the major arc $\frown AB$. In the next lesson we are going to look at the relationship between the size of the central angle and the length of the corresponding arc (which is the basis for radian measure). In the meantime, I will use the correspondence as a way to simplify my illustrations– by using an arc to indicate a central angle, I can keep the picture from getting too crowded around the center of the circle.

The Inscribed Angle Theorem

In this section we will prove the Inscribed Angle Theorem, a result which is indispensible when working with circles. I suspect that this theorem is the most elementary result of Euclidean geometry which is generally *not* known to the average calculus student. Before stating the theorem, we must define an inscribed angle, the subject of the theorem.

DEF: INSCRIBED ANGLE
If A, B, and C are all points on a circle, then $\angle ABC$ is an *inscribed angle* on that circle.

Given any inscribed angle $\angle ABC$, points A and C are the endpoints of two arcs (either a minor and a major arc or two semicircles). Excluding the endpoints, one of those two arcs will be contained in the interior of $\angle ABC$ (a homework problem). We say, then, that $\angle ABC$ is inscribed on that arc. The Inscribed Angle Theorem describes the close relationship between an inscribed angle and the central angle on the same arc.

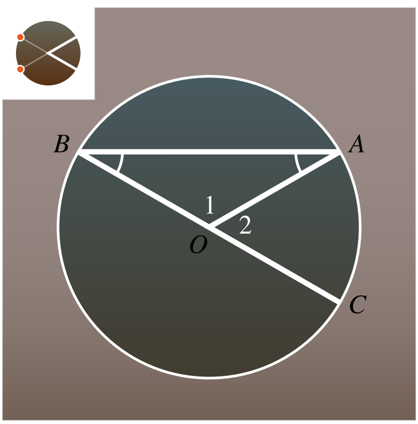


Two inscribed angles

THE INSCRIBED ANGLE THEOREM
If $\angle BAC$ is an inscribed angle on a circle with center O, then

$$(\angle ABC) = \frac{1}{2}(\angle AOC).$$

Proof. This proof is a good lesson on the benefits of starting off with an easy case. There are three parts to this proof, depending upon the location of the vertex B relative to the lines OA and OC.



Part 1. When B is the intersection of $OC\rightarrow^{op}$ *with the circle, or when B is the intersection of* $OA\ \rightarrow^{op}$ *with the circle.*

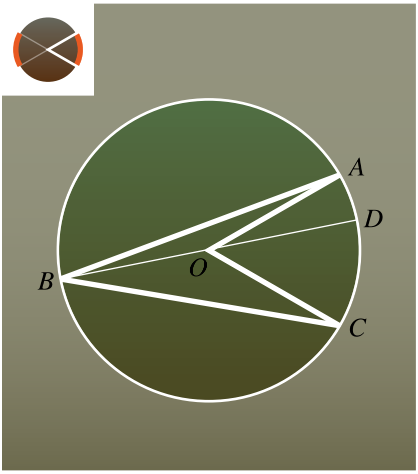
Even though we are only establishing the theorem for two very particular locations of B, this part is the key that unlocks everything else. Now, while I have given two possible locations for B, I am going to prove the result for just the first one (where B is on $OC\rightarrow^{op}$). All you have to do to prove the other part is to switch the letters A and C. Label $\angle AOB$ as $\angle 1$ and $\angle AOC$ as $\angle 2$. These angles are supplementary, so

$$(\angle 1) + (\angle 2) = 180^\circ. \quad (i)$$

The angle sum of $\triangle AOB$ is 180°, but in that triangle $\angle A$ and $\angle B$ are opposite congruent segments, so by the Isosceles Triangle Theorem they are congruent. Therefore

$$2(\angle B) + (\angle 1) = 180^\circ, \quad (ii)$$

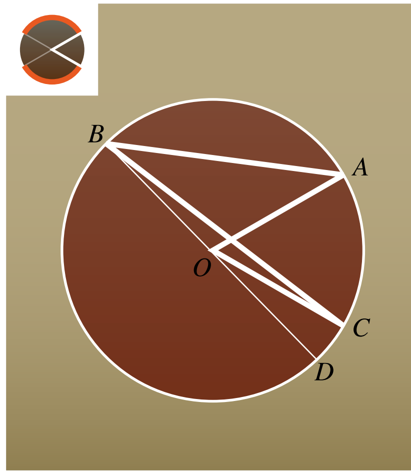
and if we subtract equation (ii) from equation (i), we get $(\angle 2) - 2(\angle B) = 0$, so $(\angle AOC) = 2(\angle ABC)$.



Part 2. When B is in the interior of $\angle AOC$, or when B is in the interior of the angle formed by $OA\!\rightarrow^{op}$ and $OC\!\rightarrow^{op}$, or when $A * O * C$.

There are three scenarios here– in the first the central angle is reflex, in the second it is proper, and in the third it is a straight angle– but the proof is the same for all of them. In each of these scenarios, the line $\leftarrow OB \rightarrow$ splits both the inscribed and the central angles. In order to identify these four angles, let me label one more point: D is the second intersection of $\leftarrow OB \rightarrow$ with the circle (so BD is a diameter of the circle). Using angle addition in conjunction with the previous results,

$$\begin{aligned}(\angle AOC) &= (\angle AOD) + (\angle DOC)\\ &= 2(\angle ABD) + 2(\angle DBC)\\ &= 2((\angle ABD) + (\angle DBC))\\ &= 2(\angle ABC).\end{aligned}$$



Part 3. When B is in the interior of the angle formed by $OA \rightarrow$ and $OC \rightarrow^{op}$, or when B is in the interior of the angle formed by $OC \rightarrow$ and $OA\!\rightarrow^{op}$.

As in the last case, label D so that BD is a diameter. The difference this time is that we need to use angle subtraction instead of angle addition. Since subtraction is a little less symmetric than addition, the two scenarios will differ slightly (in terms of lettering). In the first scenario

$$\begin{aligned}(\angle AOC) &= (\angle AOD) - (\angle DOC)\\ &= 2(\angle ABD) - 2(\angle DBC)\\ &= 2((\angle ABD) - (\angle DBC))\\ &= 2(\angle ABC).\end{aligned}$$

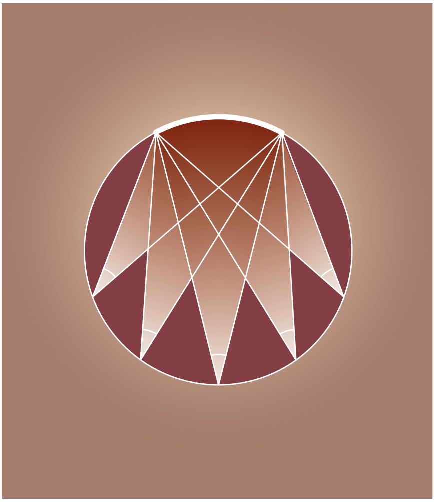
To get the second, you just need to switch A and C. □

There are two important and immediate corollaries to this theorem. First, because all inscribed angles on a given arc share the same central angle,

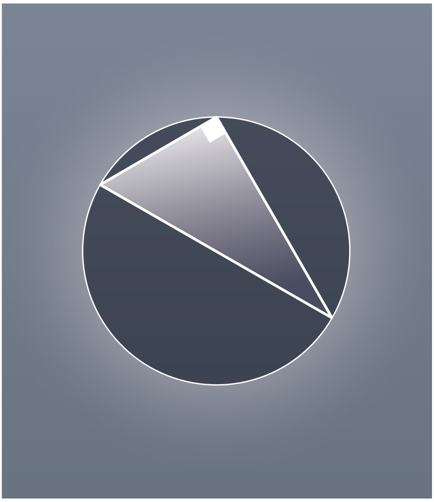
COR 1
All inscribed angles on a given arc are congruent.

Second, the special case where the central angle $\angle AOC$ is a straight angle, so that the inscribed $\angle ABC$ is a right angle, is important enough to earn its own name

THALES' THEOREM
If C is a point on a circle with diameter AB (and C is neither A nor B), then $\triangle ABC$ is a right triangle.



Five congruent angles inscribed on the same arc.



A right angle inscribed on a semicircle.

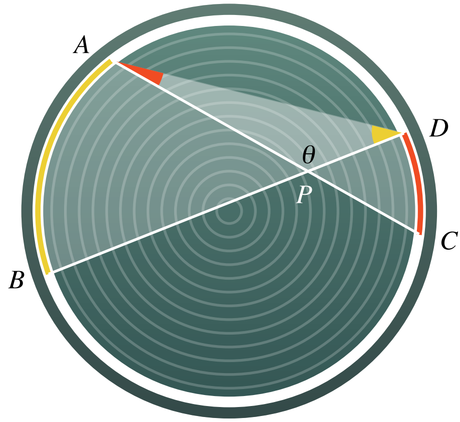
Applications of the Inscribed Angle Theorem

Using the Inscribed Angle Theorem, we can establish several nice relationships between chords, secants, and tangents associated with a circle. I will look at two of these results to end this lesson and put some more in the exercises.

THE CHORD-CHORD FORMULA

Let $\mathcal{C}$ be a circle with center O. Suppose that AC and BD are chords of this circle, and suppose further that they intersect at a point P. Label the angle of intersection, $\theta = \angle APD \simeq \angle BPC$. Then

$$(\theta) = \frac{(\angle AOD) + (\angle BOC)}{2}.$$



Proof. The angle θ is an interior angle of $\triangle APD$, so

$$(\theta) = 180^\circ - (\angle A) - (\angle D).$$

Both $\angle A$ and $\angle D$ are inscribed angles– $\angle A$ is inscribed on the arc $\smile CD$ and $\angle D$ is inscribed on the arc $\smile AB$. According to the Inscribed Angle Theorem, they are half the size of the corresponding central angles, so

$$\begin{aligned}(\theta) &= 180^\circ - \tfrac{1}{2}(\angle COD) - \tfrac{1}{2}(\angle AOB)\\ &= \tfrac{1}{2}(360^\circ - (\angle COD) - (\angle AOB)).\end{aligned}$$

This is some progress, for at least now θ is related to central angles, but alas, these are not the central angles in the formula. If we add all four central angles around O, though,

$$\begin{aligned}(\angle AOB) + (\angle BOC) + (\angle COD) + (\angle DOA) &= 360^\circ\\ (\angle BOC) + (\angle DOA) &= 360^\circ - (\angle COD) - (\angle AOB).\end{aligned}$$

Now just substitute in, and you have the formula. $\square$

According to the Chord-Chord formula, as long as the intersection point P is inside the circle, θ can be computed as the average of two central angles. What would happen if P moved outside the circle? Of course then we would not be talking about chords, since chords stop at the circle boundary, but rather the secant lines containing them.

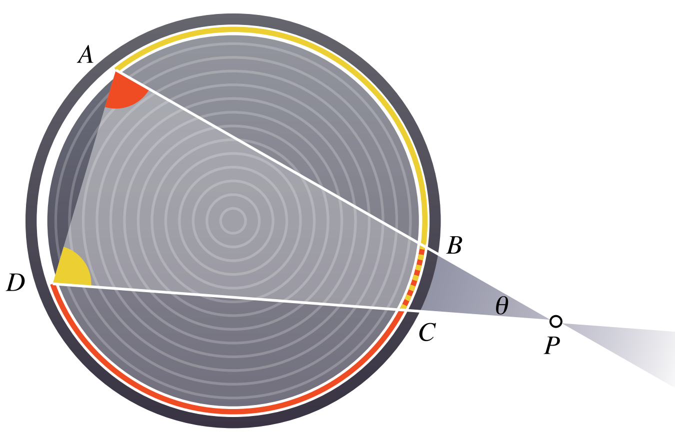
THE SECANT-SECANT FORMULA

Suppose that A, B, C, and D are points on a circle, arranged so that $\square ABCD$ is a simple quadrilateral, and that the secant lines AB and CD intersect at a point P which is outside the circle. Label the angle of intersection, $\angle APD$, as θ. If P occurs on the same side of AD as B and C, then

$$(\theta) = \frac{(\angle AOD) - (\angle BOC)}{2}.$$

If P occurs on the same side of BC as A and D, then

$$(\theta) = \frac{(\angle BOC) - (\angle AOD)}{2}.$$



Proof. There is obviously a great deal of symmetry between the two cases, so let me just address the first. The same principles apply here as in the last proof. Angle θ is an interior angle of $\triangle APD$, so

$$(\theta) = 180^\circ - (\angle A) - (\angle D).$$

Both $\angle A$ and $\angle D$ are inscribed angles– $\angle A$ is inscribed on arc $\smile BD$ and $\angle D$ is inscribed on arc $\smile AC$. We need to use the Inscribed Angle Theorem to relate these angles to central angles, and in this case, those central angles overlap a bit, so we will need to break them down further, but the rest is straightforward.

$$\begin{aligned}
(\theta) &= 180^\circ - \tfrac{1}{2}(\angle BOD) - \tfrac{1}{2}(\angle AOC) \\
&= \tfrac{1}{2}(360^\circ - (\angle BOD) - (\angle AOC)) \\
&= \tfrac{1}{2}(360^\circ - (\angle BOC) - (\angle COD) - (\angle AOB) - (\angle BOC)) \\
&= \tfrac{1}{2}([360^\circ - (\angle AOB) - (\angle BOC) - (\angle COD)] - (\angle BOC)) \\
&= \tfrac{1}{2}((\angle AOD) - (\angle BOC)).
\end{aligned}$$

□

Exercises

1. Verify that the length of a diameter of a circle is twice the radius.

2. Prove that no line is entirely contained in any circle.

3. Prove that a circle is convex. That is, prove that if points P and Q are inside a circle, then all the points on the segment PQ are inside the circle.

4. Prove that for any circle there is a triangle entirely contained in it (all the points of the triangle are inside the circle).

5. Prove that for any circle there is a triangle which entirely contains it (all the points of the circle are in the interior of the triangle).

6. In the proof that two circles intersect at most twice, I have called both

   (1) $|a-b| < c < a+b$, and (2) $c \geq a, b$ and $c < a+b$

   the Triangle Inequality conditions. Verify that the two statements are equivalent for any three positive real numbers.

7. Let $\angle ABC$ be an inscribed angle on a circle. Prove that, excluding the endpoints, exactly one of the two arcs $\smile AC$ lies in the interior of $\angle ABC$.

8. Prove the converse of Thales' theorem: if $\triangle ABC$ is a right triangle with right angle at C, then C is on the circle with diameter AB.

9. Consider a simple quadrilateral which is inscribed on a circle (that is, all four vertices are on the circle). Prove that the opposite angles of this quadrilateral are supplementary.

10. Let C be a circle and P be a point outside of it. Prove that there are exactly two lines which pass through P and are tangent to C. Let Q and R be the points of tangency for the two lines. Prove that PQ and PR are congruent.

11. The "Tangent-Tangent" formula. Let P be a point which is outside of a circle $\mathcal{C}$. Consider the two tangent lines to $\mathcal{C}$ which pass through P and let A and B be the points of tangency between those lines and the circle. Prove that

$$(\angle APB) = \frac{(\angle 1) - (\angle 2)}{2}$$

where $\angle 1$ is the reflex central angle corresponding to the major arc $\frown AB$ and $\angle 2$ is the proper central angle corresponding to the minor arc $\smile AB$.

12. Let AC and BD be two chords of a circle which intersect at a point P inside that circle. Prove that

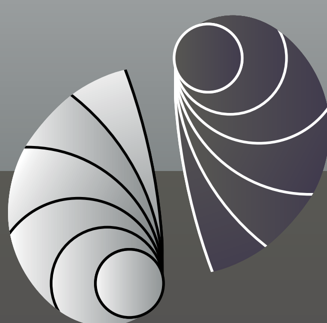
$$|AP| \cdot |CP| = |BP| \cdot |DP|.$$

References

I learned of the Chord-Chord, Secant-Secant, and Tangent-Tangent formulas in the Wallace and West book *Roads to Geometry*[1]. They use the names Two-Chord Angle Theorem, Two-Secant Angle Theorem, and Two-Tangent Angle Theorem.

[1] Edward C. Wallace and Stephen F. West. *Roads to Geometry*. Pearson Education, Inc., Upper Saddle River, New Jersey, 3rd edition, 2004.

...COMES AROUND



17 CIRCUMFERENCE.

A theorem on perimeters

In the lesson on polygons, I defined the perimeter of a polygon $\mathcal{P} = P_1 \cdots P_n$ as

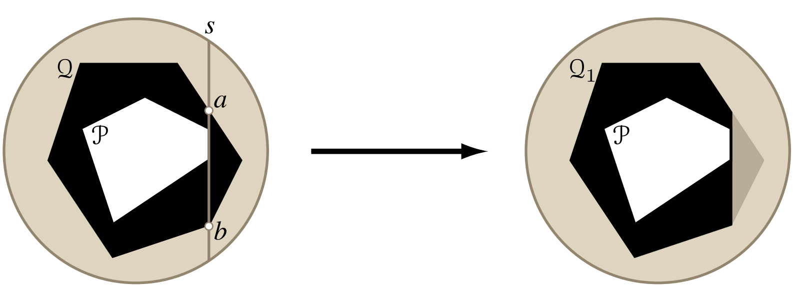
$$|\mathcal{P}| = \sum_{i=1}^{n} |P_i P_{i+1}|,$$

but I left it at that. In this lesson we are going to use perimeters of cyclic polygons to find the circumference of the circle. Along the way, I want to use the following result which compares the perimeters of two convex polygons when one is contained in the other.

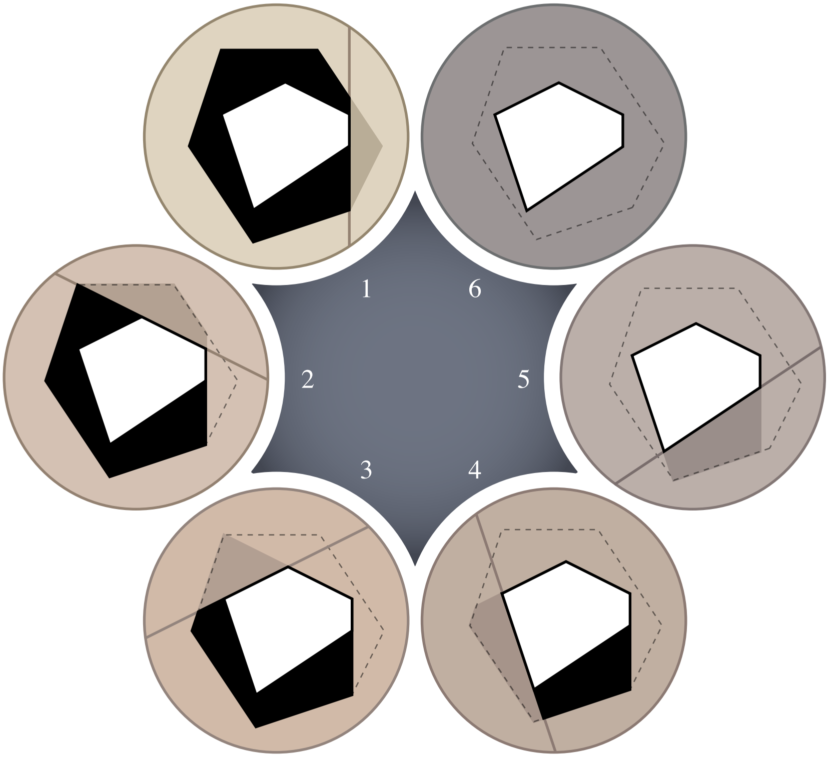
THM 1
If $\mathcal{P}$ and $\mathcal{Q}$ are convex polygons and all the points of $\mathcal{P}$ are on or inside $\mathcal{Q}$, then $|\mathcal{P}| \leq |\mathcal{Q}|$.

Proof. Some of the edges of $\mathcal{P}$ may run along the edges of $\mathcal{Q}$, but unless $\mathcal{P} = \mathcal{Q}$, at least one edge of $\mathcal{P}$ must pass through the interior of $\mathcal{Q}$. Let s be one of those interior edges. The line containing s intersects $\mathcal{Q}$ twice– call those intersections a and b– dividing $\mathcal{Q}$ into two smaller polygons which share the side ab, one on the same side of s as $\mathcal{P}$, the other on the opposite side. Essentially we want to "shave off" the part of $\mathcal{Q}$ on the opposite side, leaving behind only the polygon $\mathcal{Q}_1$ which consists of

- points of $\mathcal{Q}$ on the same side of s as $\mathcal{P}$, and
- points on the segment ab.



Shaving a polygon.



One at a time, shave the sides of the outer polygon down to the inner one.

There are two things to notice about $\mathcal{Q}_1$. First, $\mathcal{Q}_1$ and $\mathcal{P}$ have one more coincident side (the side s) than $\mathcal{Q}$ and $\mathcal{P}$ had. Second, the portions of $\mathcal{Q}$ and $\mathcal{Q}_1$ on the side of s with $\mathcal{P}$ are identical, so the segments making up that part contribute the same amount to their respective perimeters. On the other side, though, the path that $\mathcal{Q}$ takes from a to b is longer than the direct route along the segment ab of $\mathcal{Q}_1$ (because of the Triangle Inequality). Combining the two parts, that means $|\mathcal{Q}_1| \leq |\mathcal{Q}|$.

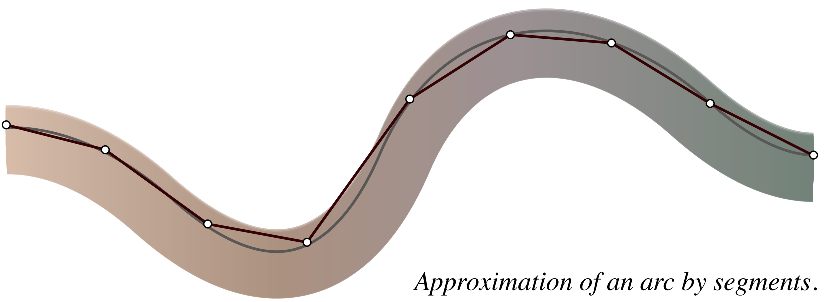
Now we can repeat this process with $\mathcal{P}$ and $\mathcal{Q}_1$, generating $\mathcal{Q}_2$ with even smaller perimeter than $\mathcal{Q}_1$ and another coincident side with $\mathcal{P}$. And again, to get $\mathcal{Q}_3$. Eventually, though, after say m steps, we run out of sides that pass through the interior, at which point $\mathcal{P} = \mathcal{Q}_m$. Then

$$|\mathcal{P}| = |\mathcal{Q}_m| \leq |\mathcal{Q}_{m-1}| \leq \cdots |\mathcal{Q}_2| \leq |\mathcal{Q}_1| \leq |\mathcal{Q}|.$$

□

Circumference

Geometers have drawn circles for a long time. I don't think it is a big surprise, then, that they would wonder about the relationship between the distance around the circle (how far they have dragged their pencil) and the radius of the circle. The purpose of this lesson is to answer that question. Our final result, the formula $C = 2\pi r$, sits right next to the Pythagorean Theorem in terms of star status, but I think it is a misunderstood celebrity. So let me be clear about what this equation is *not*. It is *not* an equation comparing two known quantities C and $2\pi r$. Instead, this equation is the way that we define the constant π. Nevertheless, the equation is saying *something* about the relationship between C and r– it is saying that the ratio of the two is a constant.



Approximation of an arc by segments.

To define the circumference of a circle, I want to take an idea from calculus– the idea of approximating a curve by straight line segments, and then refining the approximation by increasing the number of segments. In the case of a circle $\mathcal{C}$, the approximating line segments will be the edges of a simple cyclic polygon $\mathcal{P}$ inscribed in the circle. Conceptually, we will want the circumference of $\mathcal{C}$ to be bigger than the perimeter of $\mathcal{P}$. We should also expect that by adding in additional vertices to $\mathcal{P}$, we should be able to get the perimeter of $\mathcal{P}$ as close as we want to the circumference of $\mathcal{C}$. All this suggests (to me at least) that to get the circumference of $\mathcal{C}$, we need to find out how large the perimeters of inscribed polygons can be.

DEF: CIRCUMFERENCE

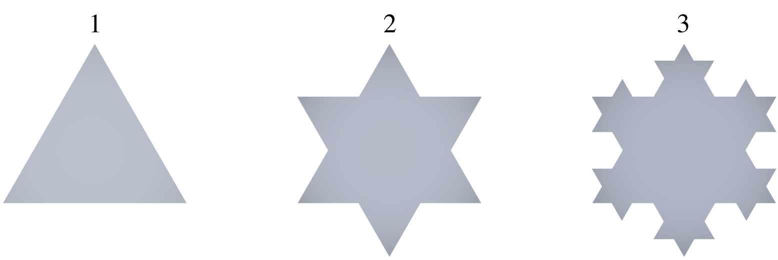
The circumference of a circle $\mathcal{C}$, written $|\mathcal{C}|$, is

$$|\mathcal{C}| = \sup\left\{ |\mathcal{P}| \,\middle|\, \mathcal{P} \text{ is a simple cyclic polygon inscribed in } \mathcal{C} \right\}.$$

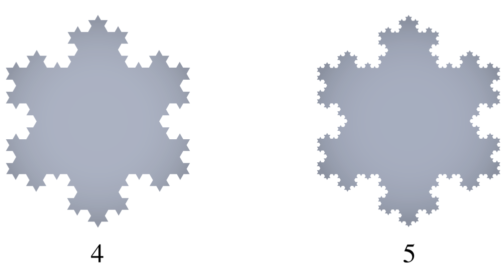
There is nothing in the definition to guarantee that this supremum exists. It is conceivable that the lengths of these approximating perimeters might just grow and grow with bound. One example of such degeneracy is given the deceptively cute name of "the Koch snowflake." Let me describe how it works. Take an equilateral triangle with sides of length one. The perimeter of this triangle is, of course, 3. Now divide each of those sides into thirds. On each middle third, build an equilateral triangle by adding two more sides; then remove the the original side. You have made a shape with $3 \cdot 4$ sides, each with a length 1/3, for a perimeter of 4. Now iterate– divide each of those sides into thirds; build equilateral triangles on each middle third, and remove the base. That will make $3 \cdot 16$ sides of length 1/9, for a perimeter of 16/3. Then $3 \cdot 64$ sides of length 1/27 for a perimeter of $64/9$. Generally, after n iterations, there are $3 \cdot 4^n$ sides of length $1/3^n$ for a total perimeter of $4^n/3^{n-1}$, and

$$\lim_{n\to\infty} \frac{4^n}{3^{n-1}} = \lim_{n\to\infty} 3\left(\frac{4}{3}\right)^n = \infty.$$

The Koch snowflake, which is the limiting shape in this process, has infinite perimeter! The first thing we need to do, then, is to make sure that circles are better behaved than this.

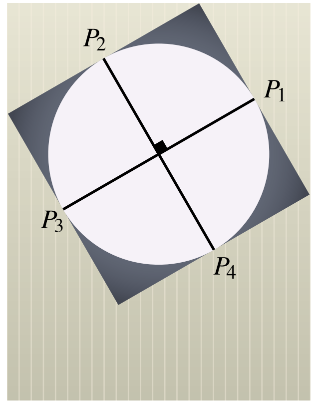


The first few steps in the construcion of the Koch snowflake.



AN UPPER BOUND FOR CIRCUMFERENCE

If $\mathcal{C}$ is a circle of radius r, then $|\mathcal{C}| \le 8r$.



Proof. The first step is to build a circumscribing square around $\mathcal{C}$– the smallest possible square that still contains $\mathcal{C}$. Begin by choosing two perpendicular diameters d_1 and d_2. Each will intersect $\mathcal{C}$ twice, for a total of four intersections, P_1, P_2, P_3, and P_4. For each i between one and four, let t_i be the tangent line to $\mathcal{C}$ at P_i. These tangents intersect to form the circumscribing square. The length of each side of the square is equal to the diameter of $\mathcal{C}$, so the perimeter of the square is $4 \cdot 2r = 8r$.

Now we turn to the theorem we proved to start this lesson. Each simple cyclic polygon inscribed in $\mathcal{C}$ is a convex polygon contained in the circumscribing square. Therefore the perimeter of any such approximating polygon is bounded above by $8r$. Remember that we have defined $|\mathcal{C}|$ to be the supremum of all of these approximating perimeters, so it cannot exceed $8r$ either. □

Now that we know that any circle does have a circumference, the next step is to find a way to calculate it. The key to that is the next theorem.

CIRCUMFERENCE/RADIUS

The ratio of the circumference of a circle to its radius is a constant.

Proof. Let's suppose that this ratio is not a constant, so that there are two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with centers O_1 and O_2 and radii r_1 and r_2, but with unequal ratios

$$|\mathcal{C}_1|/r_1 > |\mathcal{C}_2|/r_2.$$

As we have defined circumference, there are approximating cyclic polygons to $\mathcal{C}_1$ whose perimeters are arbitrarily close to its circumference. In particular, there has to be some approximating cyclic polygon $\mathcal{P} = P_1P_2 \ldots P_n$ for $\mathcal{C}_1$ so that

$$|\mathcal{P}|/r_1 > |\mathcal{C}_2|/r_2.$$

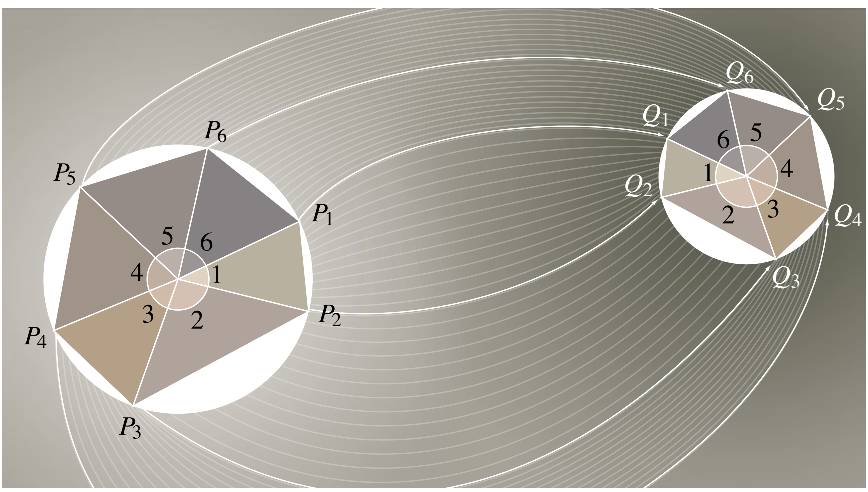
The heart of the contradiction is that we can build a cyclic polygon $\mathcal{Q}$ on $\mathcal{C}_2$ which is similar to $\mathcal{P}$ (intuitively, we just need to scale $\mathcal{P}$ so that it fits in the circle). The construction is as follows

1. Begin by placing a point Q_1 on circle $\mathcal{C}_2$.

2. Locate Q_2 on $\mathcal{C}_2$ so that $\angle P_1O_1P_2$ is congruent to $\angle Q_1O_2Q_2$ (there are two choices for this).

3. Locate Q_3 on $\mathcal{C}_2$ and on the opposite side of O_2Q_2 from Q_1 so that $\angle P_2O_1P_3 \simeq \angle Q_2O_2Q_3$.

4. Continue placing points on $\mathcal{C}_2$ in this fashion until Q_n has been placed to form the polygon $\mathcal{Q} = Q_1Q_2\dots Q_n$.



Then

$$\frac{|O_2Q_i|}{|O_1P_i|} = \frac{r_2}{r_1} = \frac{|O_2Q_{i+1}|}{|O_1P_{i+1}|} \quad \& \quad \angle Q_iO_2Q_{i+1} \simeq \angle P_iO_1P_{i+1},$$

so by S·A·S similarity, $\triangle Q_iO_2Q_{i+1} \sim \triangle P_iO_1P_{i+1}$. That gives us the ratio of the third sides of the triangle as $|Q_iQ_{i+1}|/|P_iP_{i+1}| = r_2/r_1$ and so we can describe the perimeter of $\mathcal{Q}$ as

$$|\mathcal{Q}| = \sum_{i=1}^{n} |Q_iQ_{i+1}| = \sum_{i=1}^{n} \frac{r_2}{r_1}|P_iP_{i+1}| = \frac{r_2}{r_1}\sum_{i=1}^{n} |P_iP_{i+1}| = \frac{r_2}{r_1}|\mathcal{P}|.$$

Here's the problem. That would mean that

$$\frac{|\mathcal{Q}|}{r_2} = \frac{|\mathcal{P}|}{r_1} > \frac{|\mathcal{C}_2|}{r_2}$$

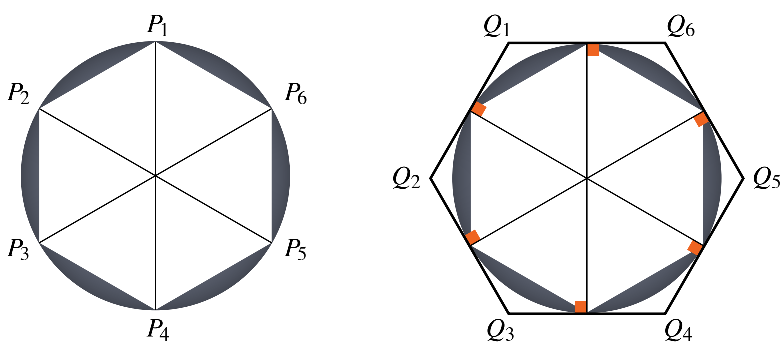
so $|\mathcal{Q}| > |\mathcal{C}_2|$ when the circumference of $\mathcal{C}_2$ is supposed to be greater than the perimeter of any of the approximating cyclic polygons. □

DEF: π

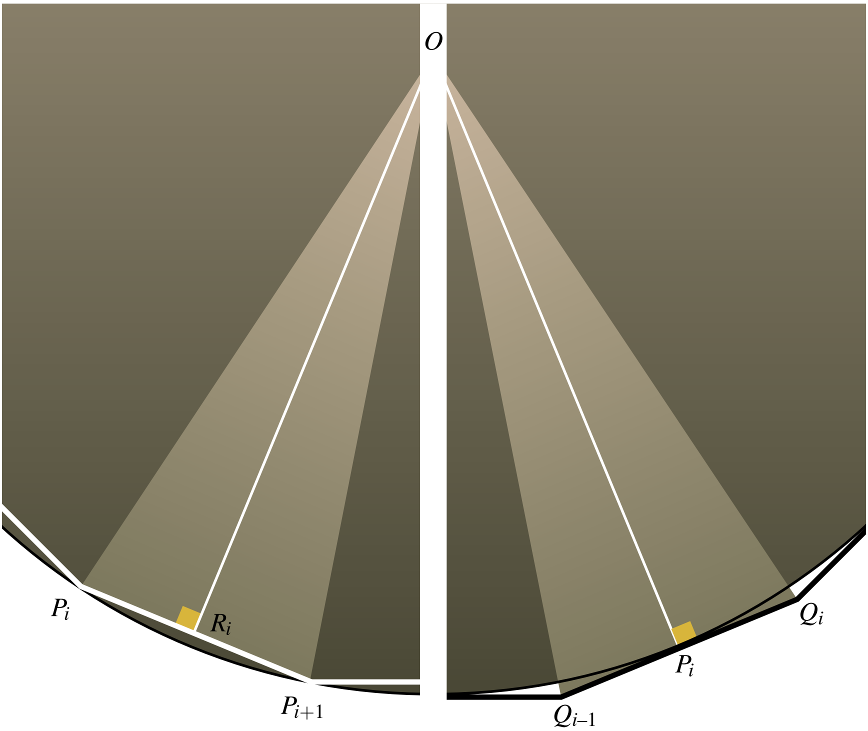
The constant π is the ratio of the circumference of a circle to its diameter

$$\pi = \frac{|\mathcal{C}|}{2r}.$$

The problem with this definition of circumference, and consequently this definition of π, is that it depends upon a supremum, and supremums are ungainly and difficult to maneuver. A limit is considerably more nimble. Fortunately, this particular supremum can be reached via the perimeters of a sequence of regular polygons as follows. Arrange n angles each measuring $360°/n$ around the center of any circle $\mathcal{C}$. The rays of those angles intersect $\mathcal{C}$ n times, and these points P_i are the vertices of a regular n-gon, $\mathcal{P}_n = P_1P_2\ldots P_n$. The tangent lines to $\mathcal{C}$ at the neighboring points P_i and P_{i+1} intersect at a point Q_i. Taken together, these n points are the vertices of another regular n-gon $\mathcal{Q}_n = Q_1Q_2\ldots Q_n$. The polygon $\mathcal{P}_n$ is just one of the many cyclic polygons inscribed in $\mathcal{C}$ so $|\mathcal{P}_n| \le |\mathcal{C}|$. The polygon $\mathcal{Q}_n$ circumscribes $\mathcal{C}$, and every cyclic polygon inscribed on $\mathcal{C}$ lies inside $\mathcal{Q}_n$, so $|\mathcal{Q}_n| \ge |\mathcal{C}|$.



Regular inscribed and circumscribing hexagons.



The lower bound prescribed by $\mathcal{P}_n$. Each $OQ_i \rightarrow$ is a perpendicular bisector of P_iP_{i+1}, intersecting it at a point R_i and dividing $\triangle OP_iP_{i+1}$ in two. By the H·L congruence theorem for right triangles, those two parts, $\triangle OR_iP_i$ and $\triangle OR_iP_{i+1}$, are congruent. That means that $\mathcal{P}_n$ is built from $2n$ segments of length $|P_iR_i|$. Now

$$\sin(360^\circ/2n) = \frac{|P_iR_i|}{r}$$
$$\implies |P_iR_i| = r\sin(360^\circ/2n)$$

so

$$|\mathcal{P}_n| = 2nr\sin(360^\circ/2n).$$

The upper bound prescribed by $\mathcal{Q}_n$. Each $OP_i \rightarrow$ is a perpendicular bisector of $Q_{i-1}Q_i$, intersecting it at P_i and dividing $\triangle OQ_{i-1}Q_i$ in two. By S·A·S, the two parts, $\triangle OP_iQ_{i-1}$ and $\triangle OP_iQ_i$, are congruent. That means $\mathcal{Q}_n$ is built from $2n$ segments of length $|P_iQ_i|$. Now

$$\tan(360^\circ/2n) = |P_iQ_i|/r$$
$$\implies |P_iQ_i| = r\tan(360^\circ/2n)$$

so

$$|\mathcal{Q}_n| = 2nr\tan(360^\circ/2n).$$

Let's compare $|\mathcal{P}_n|$ and $|\mathcal{Q}_n|$ as n increases (the key to this calculation is that as x approaches zero, $\cos(x)$ approaches one):

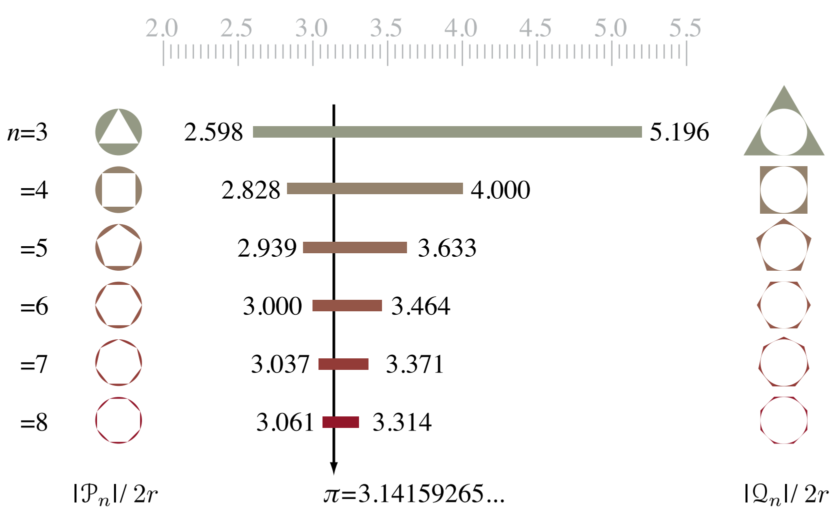
$$\begin{aligned}
\lim_{n\to\infty} |\mathcal{Q}_n| &= \lim_{n\to\infty} 2nr\tan(360^\circ/2n) \\
&= \lim_{n\to\infty} \frac{2nr\sin(360^\circ/2n)}{\cos(360^\circ/2n)} \\
&= \frac{\lim_{n\to\infty} 2nr\sin(360^\circ/2n)}{\lim_{n\to\infty}\cos(360^\circ/2n)} \\
&= \lim_{n\to\infty} 2nr\sin(360^\circ/2n)/1 \\
&= \lim_{n\to\infty} |\mathcal{P}_n|.
\end{aligned}$$

Since $|\mathcal{C}|$ is trapped between $|\mathcal{P}_n|$ and $|\mathcal{Q}_n|$ for all n, and since these are closing in upon the same number as n goes to infinity, $|\mathcal{C}|$ must also be approaching this number. That gives a more comfortable equation for circumference as

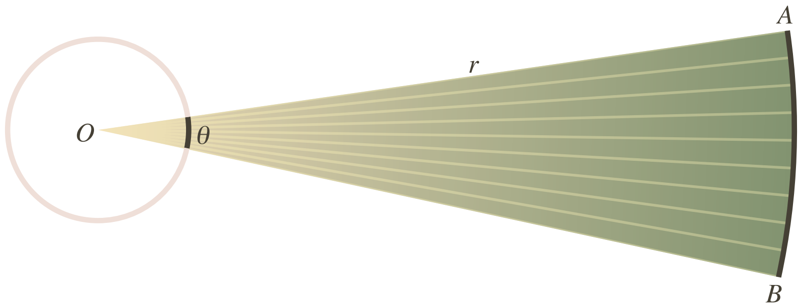
$$|\mathcal{C}| = \lim_{n\to\infty} 2nr\sin(360^\circ/2n),$$

and since $|\mathcal{C}| = 2\pi r$, we can disentangle a definition of π as

$$\pi = \lim_{n\to\infty} n\sin(360^\circ/2n).$$



Upper and lower bounds for π.



Lengths of arcs and radians.

It doesn't take much modification to get a formula for a length of arc. The 360° in the formula for $|\mathcal{C}|$ is the measure of the central angle corresponding to an arc that goes completely around the circle. To get the measure of any other arc, we just need to replace the 360° with the measure of the corresponding central angle.

LENGTHS OF CIRCULAR ARCS
If $\smile AB$ is the arc of a circle with radius r, and if θ is the measure of the central angle $\angle AOB$, then

$$|\smile AB| = \frac{\pi}{180^\circ}\theta \cdot r.$$

Proof. To start, replace the 360° in the circumference formula with θ:

$$|\smile AB| = \lim_{n\to\infty} 2nr\sin(\theta/2n) = 2r \cdot \lim_{n\to\infty} n\sin(\theta/2n).$$

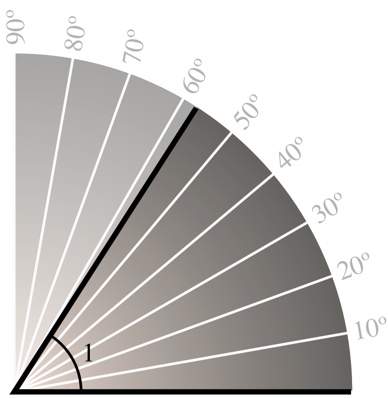
This limit is clearly related to the one that defines π. I want to absorb the difference between the two into the variable via the substitution $n = m \cdot \theta/360^\circ$. Note that as n approaches infinity, m will as well, so

$$\begin{aligned}
|\smile AB| &= 2r \cdot \lim_{m\to\infty} \frac{m\cdot\theta}{360^\circ}\sin\left(\frac{\theta}{2m\cdot\theta/360^\circ}\right) \\
&= \frac{2r\theta}{360^\circ} \cdot \lim_{m\to\infty} m\sin(360^\circ/2m) \\
&= \frac{\theta}{180^\circ} r\pi.
\end{aligned}$$

□

There is one more thing to notice before the end of this lesson. This arc length formula provides a most direct connection between angle measure (of the central angle) and distance (along the arc). And yet, the $\frac{\pi}{180^\circ}$ factor in that formula suggests that distance and the degree measurement system are a little out of sync with one another. This can be fixed by modernizing our method of angle measurement. The preferred angle measurement system, and the one that I will use from here on out, is *radian* measurement.

DEF: RADIAN
One radian is $\pi/180^\circ$.



One radian is approximately 57.296°.

The measure of a straight angle is π radians. The measure of a right angle is $\pi/2$ radians. One complete turn of the circle is 2π radians. If $\theta = (\angle AOB)$ is measured in radians, then

$$|\smile AB| = r \cdot \theta.$$

References

The Koch snowflake is an example of a fractal. Gerald Edgar's book *Measure, Topology, and Fractal Geometry* [1], deals with these objects and their measures.

[1] Gerald A. Edgar. *Measure, Topology, and Fractal Geometry*. Springer-Verlag, New York, 1st edition, 1990.

Exercises

1. Let A and B be points on a circle $\mathcal{C}$ with radius r. Let θ be the measure of the central angle corresponding to the minor arc (or semicircle) $\smile AB$. What is the relationship (in the form of an equation) between θ, r, and $|AB|$?

2. Let AB be a diameter of a circle $\mathcal{C}$, and let P be a point on AB. Let $\mathcal{C}_1$ be the circle with diameter AP and let $\mathcal{C}_2$ be the circle with diameter BP. Show that the sum of the circumferences of $\mathcal{C}_1$ and $\mathcal{C}_2$ is equal to the circumference of $\mathcal{C}$ (the shape formed by the three semicircles on one side of AB is called an *arbelos*).

3. In the construction of the Koch snowflake, the middle third of each segment is replaced with two-thirds of an equilateral triangle. Suppose, instead, that middle third was replaced with three of the four sides of a square. What is the perimeter of the n-th stage of this operation? Would the limiting perimeter still be infinite?

4. This problem deals with the possibility of angle measurement systems other than degrees or radians. Let $\mathcal{A}$ be the set of angles in the plane. Consider a function

$$\star : \mathcal{A} \to (0,\infty) : \angle A \to (\angle A)^\star$$

which satisfies the following properties

(1) if $\angle A \simeq \angle B$, then $(\angle A)^\star = (\angle B)^\star$

(2) if D is in the interior of $\angle ABC$, then

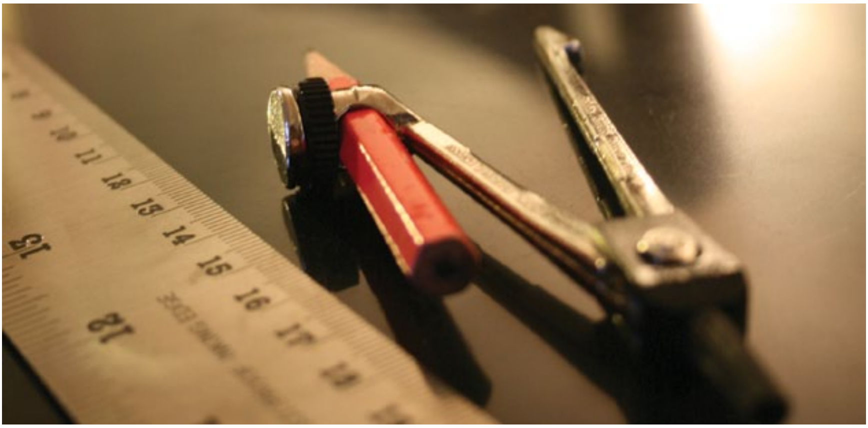
$$(\angle ABC)^\star = (\angle ABD)^\star + (\angle DBC)^\star.$$

Prove that the $\star$ measurement system is a constant multiple of the degree measurement system (or, for that matter, the radian measurement system). That is, prove that there is a $k > 0$ such that for all $\angle A \in \mathcal{A}$,

$$(\angle A)^\star = k \cdot (\angle A).$$

18 THE BLANK CANVAS AWAITS

EUCLIDEAN CONSTRUCTIONS



This lesson is a diversion from our projected path, but I maintain that it is a pleasant and worthwhile diversion. We get a break from the heavy proofs, and we get a much more tactile approach to the subject. I have found that compass and straight edge constructions serve as a wonderful training ground for the rigors of mathematics without the tricky logical pitfalls of formal proof. In my geometry classes, I often don't have time to prove many of the really neat Euclidean results that we will see in the next few lessons, but I have found that I can use compass and straight edge constructions to present the theorems in an sensible way.

Now kindly rewind all the way back to Lesson 1, when I talked briefly about Euclid's postulates. In particular, I want you to look at the first three

P1 To draw a straight line from any point to any point.

P2 To produce a finite straight line continuously in a straight line.

P3 To describe a circle with any center and distance.

Back then, I interpreted these postulates as claims of existence (of lines and circles). Consider instead a more literal reading: they are not claiming the existence of objects, but rather telling us that we can *make* them. This lesson is dedicated to doing just that: constructing geometric objects using two classical tools, a compass and a straight edge. The compass makes circles and arcs, and the straight edge makes segments, rays, and lines. Together they make the kinds of shapes that Euclid promised in his postulates.

The straight edge

The straight edge is a simple tool– it is just something that can draw lines. In all likelihood, your straight edge will be a ruler, and if so, you need to be aware of the key distinction between a ruler and a straight edge. Unlike a ruler, a straight edge has no markings (nor can you add any). Therefore, you cannot measure distance with it. But a straight edge *can* do the following :

– draw a segment between two points;
– draw a ray from a point through another point;
– draw a line through two points;
– extend a segment to either a ray or the line containing it;
– extend a ray to the line containing it.

The compass

Not to be confused with the ever-northward-pointing navigational compass, the compass of geometry is a tool for creating a circle. More precisely, a compass can do the following:

○ given two distinct points P and Q, draw the circle centered at P which passes through Q;
○ given points P and Q on a circle with a given center R, draw the arc $\smile PQ$.

You could make a simple compass by tying a pencil to a piece of string, but it would be pretty inaccurate. The metal compasses of my youth (such as the one pictured) are more precise instruments, but alas double as weaponry in the hands of some mischievous rascals. The plastic compasses that are now the norm in many schools are an adequate substitute until they fall apart, usually about halfway through the lesson.

Let me give a warning about something a compass cannot do (at least not "out of the box"). A common temptation is to try to use the compass to transfer distance. That is, to draw a circle of a certain radius, lift up the compass and move it to another location, then place it back down to draw another circle with the same radius. That process effectively transfers a distance (the radius) from one location to another, and so is a convenient way to construct a congruent copy of one segment in another location. It is a simple enough maneuver, but the problem is that according to the classical rules of the game a compass does not have this transfer ability.

The classical compass is "collapsing", meaning that as soon as it is used to create a circle, it falls apart (in this way, I guess the classical compass does resemble those shoddy plastic ones). We will soon see that the two types of compasses are *not* fundamentally different, and therefore that the non-collapsing feature is actually only a convenience. Once we have shown that, I will have no qualms about using a non-collapsing compass when it will streamline the construction process. Until then, distance transfer using a compass is off-limits.

The digital compass and straight edge

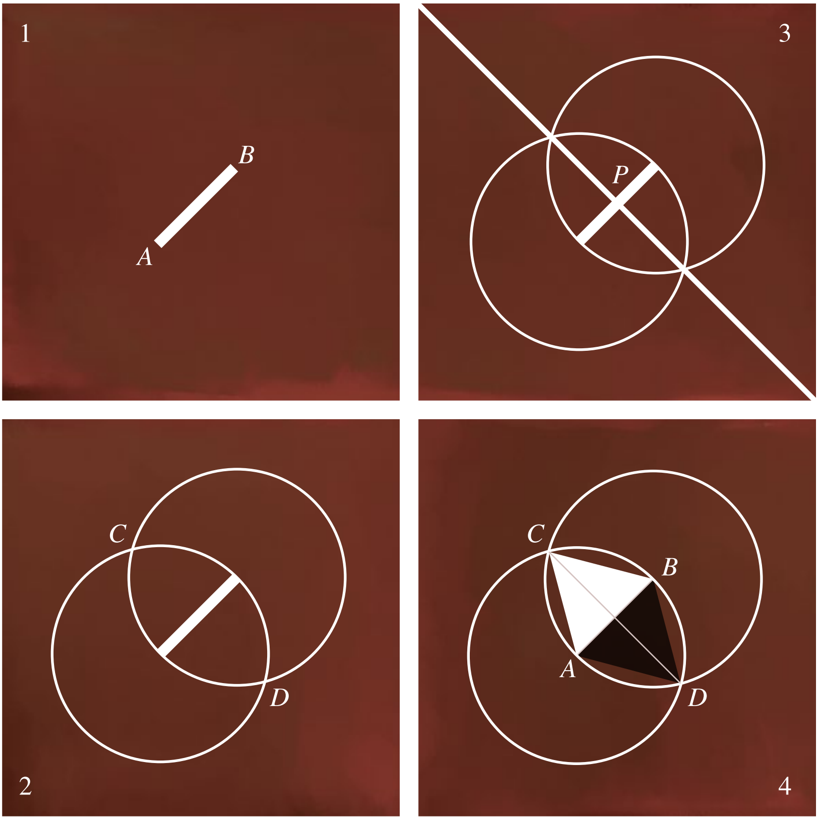
There are several good computer programs that will allow you to build these constructions digitally (though I won't formally endorse a particular one). There are both advantages and disadvantages to the digital approach. At the risk of sounding like a mystic, I believe that drawing lines and circles on a real piece of paper with a real pencil links you to a long, beautiful tradition in a way that no computer experience can. For more complicated constructions, though, the paper and pencil approach gets really messy. In addition, a construction on paper is static, while computer constructions are dynamic– you can drag points around and watch the rest of the construction adjust accordingly. Often that dynamism really reveals the power of the theorems in a way that no single static image ever could. I would recommend that you try to make a few of the simpler constructions the old-fashioned way, with pencil and paper. And I would recommend that you try a few of the more complicated constructions with the aid of a computer.

A little advice

1. It is easier to draw than to erase.

2. Lines are infinite, but your use for them may not be– try not to draw more of the line than is needed. Similarly, if you only need a small arc of a circle, there is little point in drawing the whole thing.

3. To the extent that you can plan ahead, you can build your construction so that it is neither too big nor too small. The Euclidean plane is infinite, but your piece of paper is not. At the other extreme, your real world compass likely will not function well below a certain radius.



The perpendicular bisector

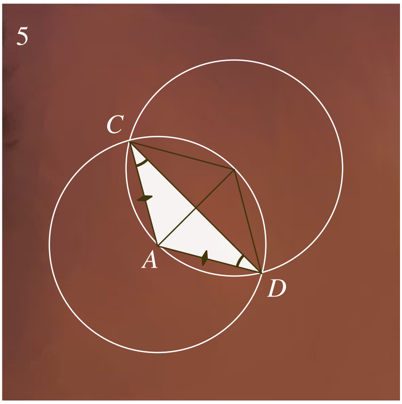
1 Begin with a segment AB.

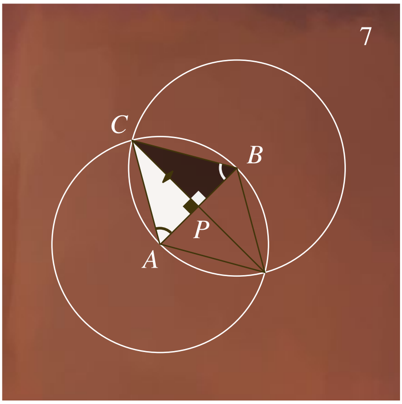
2 With the compass construct two circles: one centered at A which passes through B and one centered at B which passes through A. These circles intersect twice, at C and D, once on each side of AB.

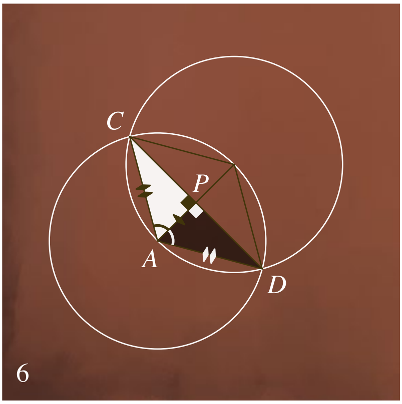
3 Use the straight edge to draw the line $\leftarrow CD \rightarrow$. That line is the perpendicular bisector of AB, and its intersection P with AB is the midpoint of AB.

Perhaps some justification of the last statement is in order. Observe the following.

4 That $\triangle ABC$ and $\triangle ABD$ are equilateral, and since they share a side, are congruent.



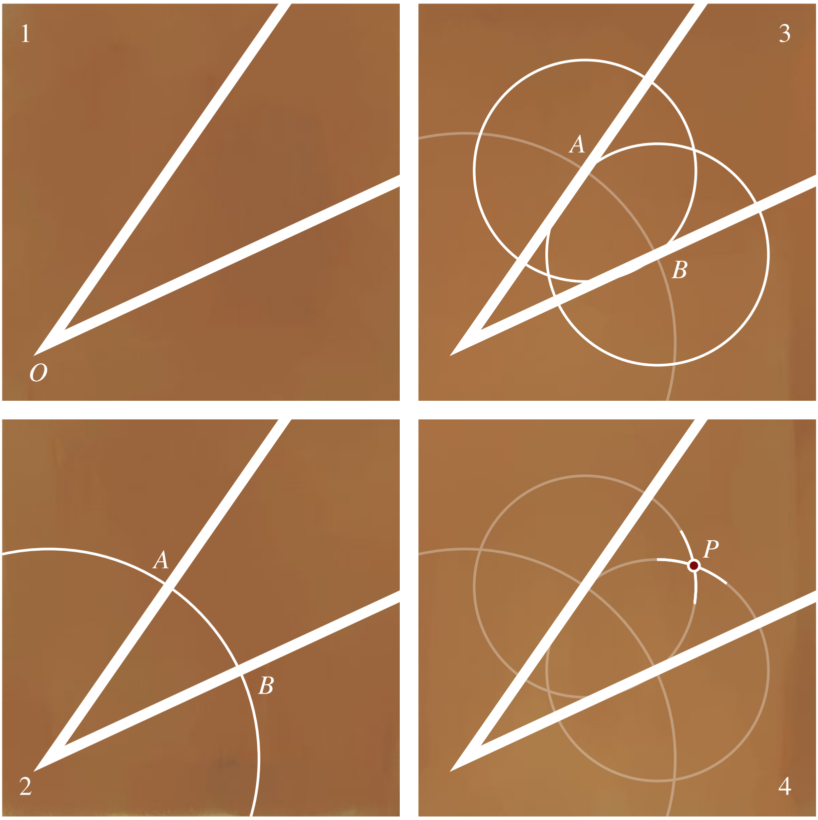




5 That $\triangle ACD$ is isosceles, so the angles opposite its congruent sides, $\angle ACD$ and $\angle ADC$, are congruent.

6 S·A·S: That $\triangle ACP$ and $\triangle ADP$ are congruent. This means $\angle APC$ is congruent to its own supplement, and so is a right angle. That handles the first part of the claim: CD is perpendicular to AB.

7 Continuing, $\angle APC$ and $\angle BPC$ are right angles. By A·A·S, $\triangle APC$ and $\triangle BPC$ are congruent and so $AP \simeq BP$. That means P has to be the midpoint of AB.



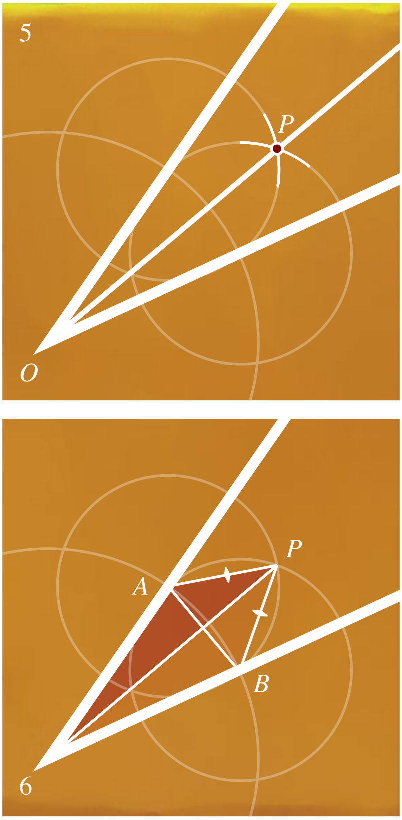
The bisector of an angle

1 Begin with an angle whose vertex is O.

2 Draw a circle centered at O, and mark where it intersects the rays that form the angle as A and B.

3 Draw two circles– one centered at A passing through B, and one centered at B passing through A.

4 Label their intersection as P.

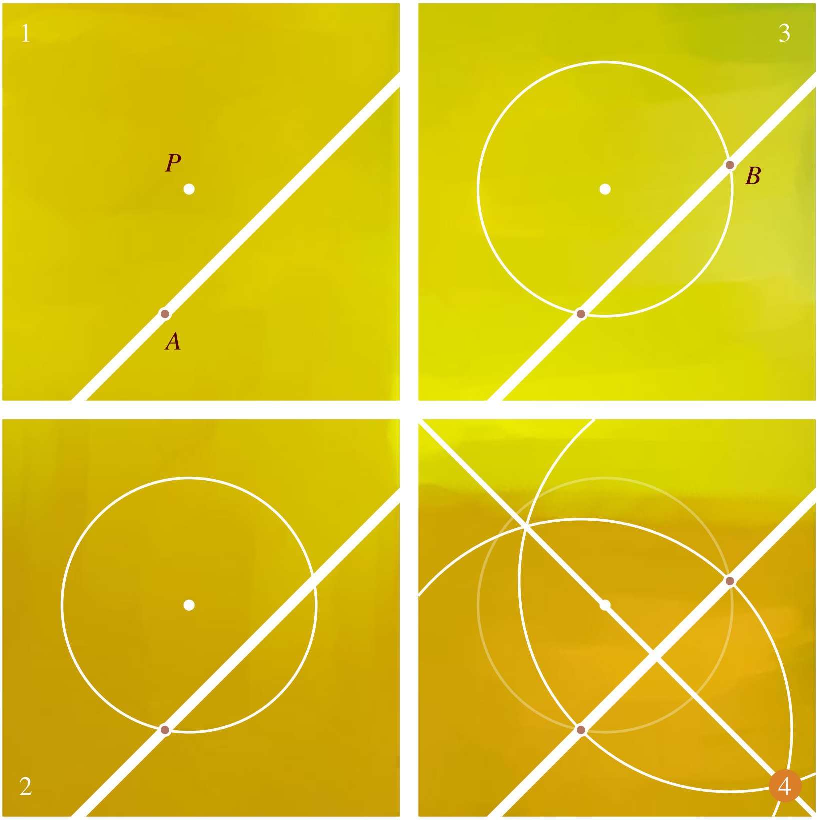


5 Draw the ray $OP\rightarrow$. It is the bisector of $\angle AOB$.

6 The justification is easier this time. You see,

$$AP \simeq AB \simeq BP$$

so by S·S·S, $\triangle OAP \simeq \triangle OBP$. Now match up the congruent interior angles, and $\angle AOP \simeq \angle BOP$.



The perpendicular to a line ℓ through a point P.

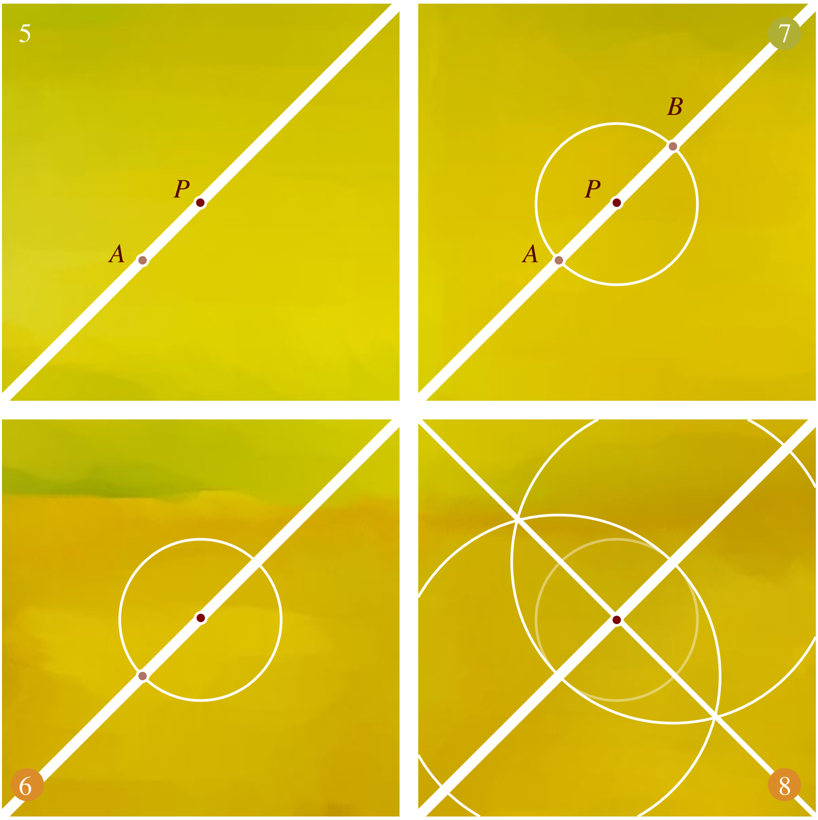
Case 1: if P is not on ℓ

1 Mark a point A on ℓ.

2 Draw the circle centered at P and passing through A.

3 If this circle intersects ℓ only once (at P), then ℓ is tangent to the circle and AP is the perpendicular to ℓ through P (highly unlikely). Otherwise, label the second intersection B.

4 Use the previous construction to find the perpendicular bisector to AB. This is the line we want.



Case 2: if P is on ℓ

5 Mark a point A on ℓ other than P.

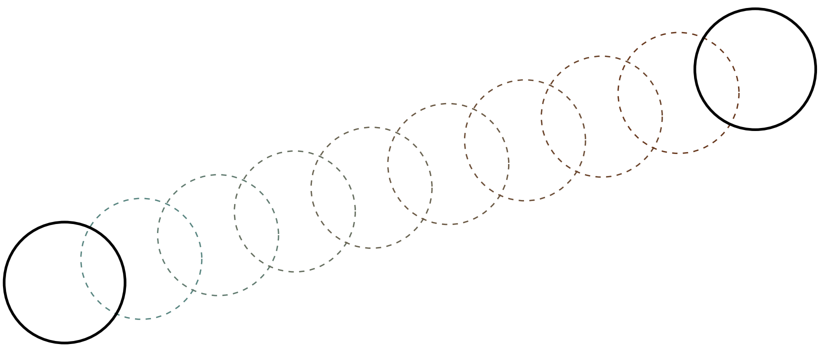
6 Draw the circle centered at P passing through A.

7 Mark the second intersection of this circle with ℓ as B.

8 Use the previous construction to find the perpendicular bisector to AB. This is the line we want.

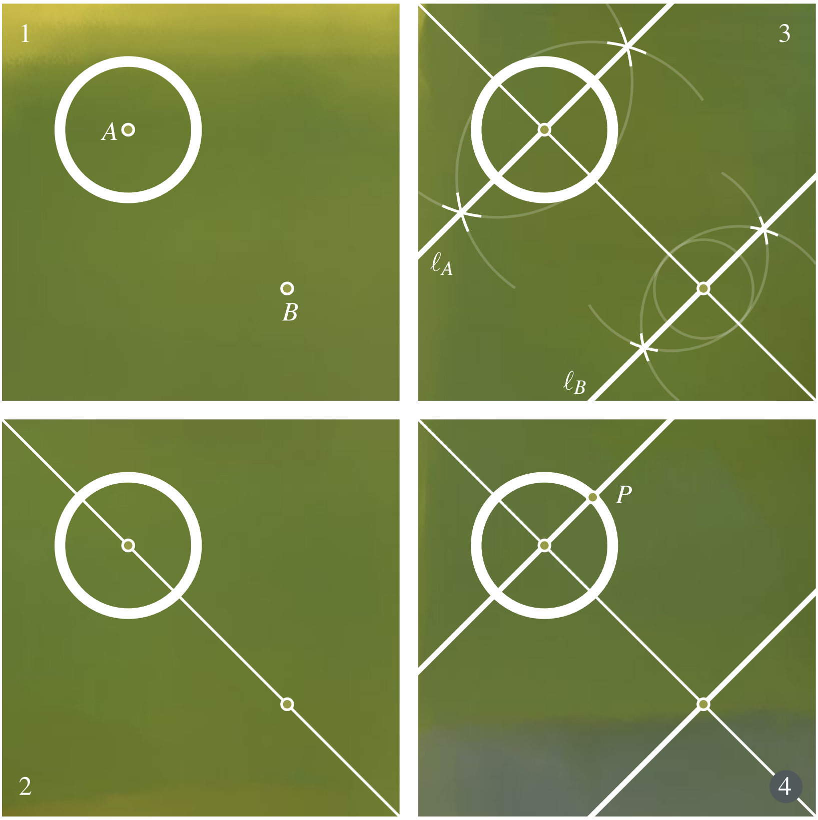
Again, there may be some question about why these constructions work. This time I am going to leave the proof to you.

Once you know how to construct perpendicular lines, constructing parallels is straightforward: starting from any line, construct a perpendicular, and then a perpendicular to that. According to the Alternate Interior Angle Theorem, the result will be parallel to the initial line. Such a construction requires quite a few steps, though, and drawing parallels feels like it should be a fairly simple procedure. As a matter of fact, there is a quicker way, but it requires a non-collapsing compass. So it is now time to look into the issue of collapsing versus non-collapsing compasses.



Collapsing v. non-collapsing

The apparent difference between a collapsing and a non-collapsing compass is that with a non-collapsing compass, we can draw a circle, move the compass to another location, and draw another circle of the same size. In effect, the non-collapsing compass becomes a mechanism for relaying information about size from one location in the plane to another. As I mentioned at the start of this lesson, the official rulebook does not permit a compass to retain and transfer that kind of information. The good news is that, in spite of this added feature, a non-collapsing compass is not any more powerful than a collapsing one. Everything that can be constructed with a non-collapsing compass can also be constructed with a collapsing one. The reason is simple: a collapsing compass can also transfer a circle from one location to another– it just takes a few more steps.



1 Begin with a circle $\mathcal{C}$ with center A. Suppose we wish to draw another circle of the same size, this time centered at a point B.

2 Construct the line $\overleftrightarrow{AB}$.

3 Construct two lines perpendicular to $\overleftrightarrow{AB}$: ℓ_A through A and ℓ_B through B.

4 Now ℓ_A intersects $\mathcal{C}$ twice: identify one point of intersection as P.

5 Construct the line ℓ_P which passes through P and is perpendicular to ℓ_A.

6 This line intersects ℓ_B. Identify the intersection of ℓ_P and ℓ_B as Q.

7 Now A, B, P, and Q are the four corners of a rectangle. The opposite sides AP and BP must be congruent. So finally,

8 Construct the circle with center B which passes through Q. This circle has the same radius as $\mathcal{C}$.

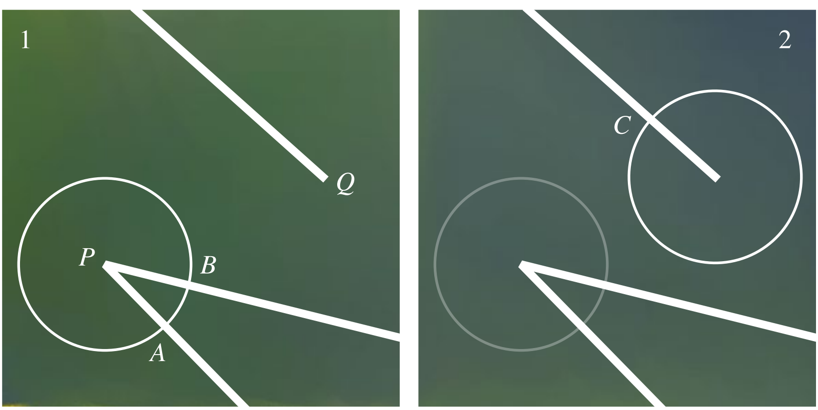
This means that a collapsing compass can do all the same things a non-collapsing compass can. From now on, let's assume that our compass has the non-collapsing capability.

Transferring segments

Given a segment AB and a ray r whose endpoint is C, it is easy to find the point D on r so that $CD \simeq AB$. Just construct the circle centered at A with radius AB, and then (since the compass is non-collapsing) move the compass to construct a circle centered at C with the same radius. The intersection of this circle and r is D.

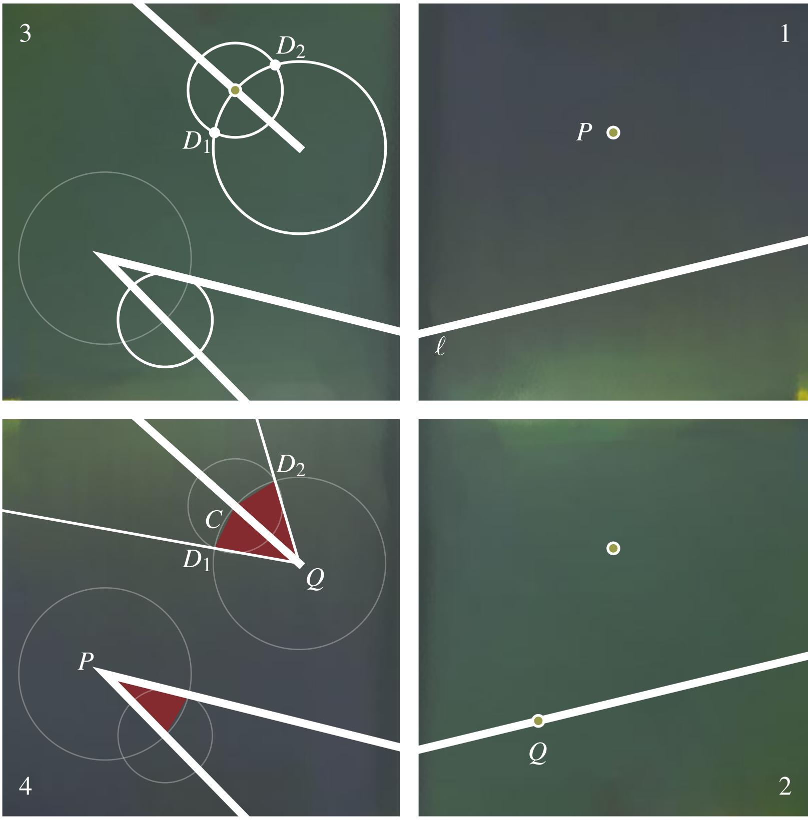
Transferring angles

Transferring a given angle to a new location is a little more complicated. Suppose that we are given an angle with vertex P and a ray r with endpoint Q, and that we want to build congruent copies of $\angle P$ off of r (there are two– one on each side of r).



1 Draw a circle with center P, and label its intersections with the two rays of $\angle P$ as A and B.

2 Using the non-collapsing compass, transfer this circle to one that is centered at Q. Call it $\mathcal{C}$ and label its intersection with r as C.



3 Draw another circle, this time one centered at A which passes through B. Then transfer it to one centered at C. The resulting circle will intersect $\mathcal{C}$ twice, once on each side of r. Label the intersection points as D_1 and D_2.

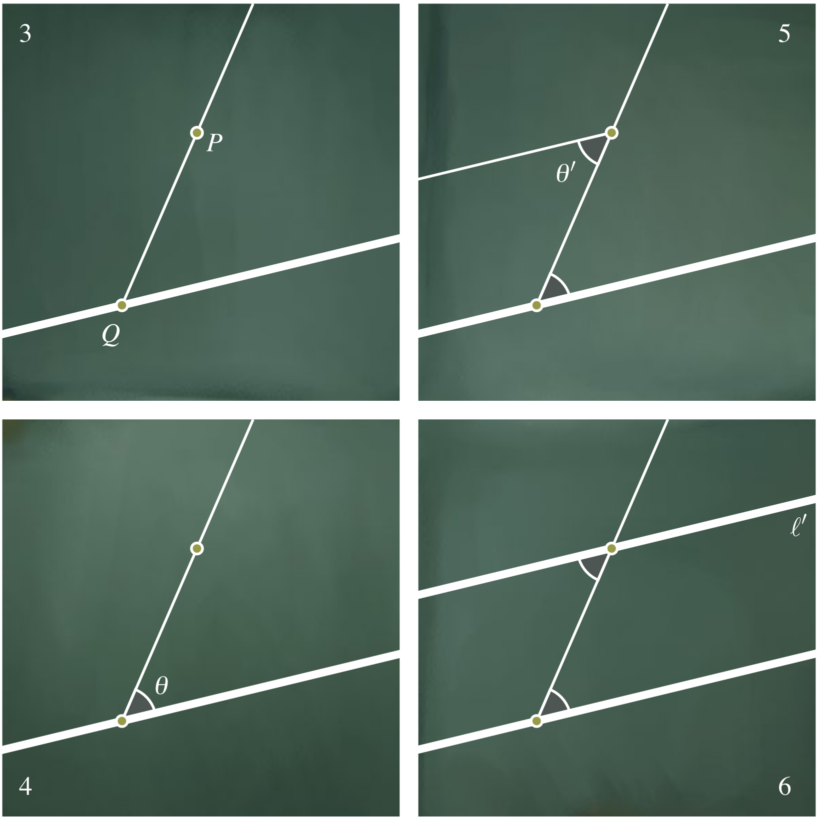
4 By S·S·S, all three of the triangles, $\triangle PAB$, $\triangle PD_1C$, and $\triangle PD_2C$ are congruent. Therefore

$$\angle D_1QC \simeq \angle P \simeq \angle D_2QC.$$

The parallel to a line through a point

1 With a non-collapsing compass and angle transfer, we can now draw parallels the "easy" way. Start with a line ℓ, and a point P which is not on that line.

2 Mark a point Q on ℓ.

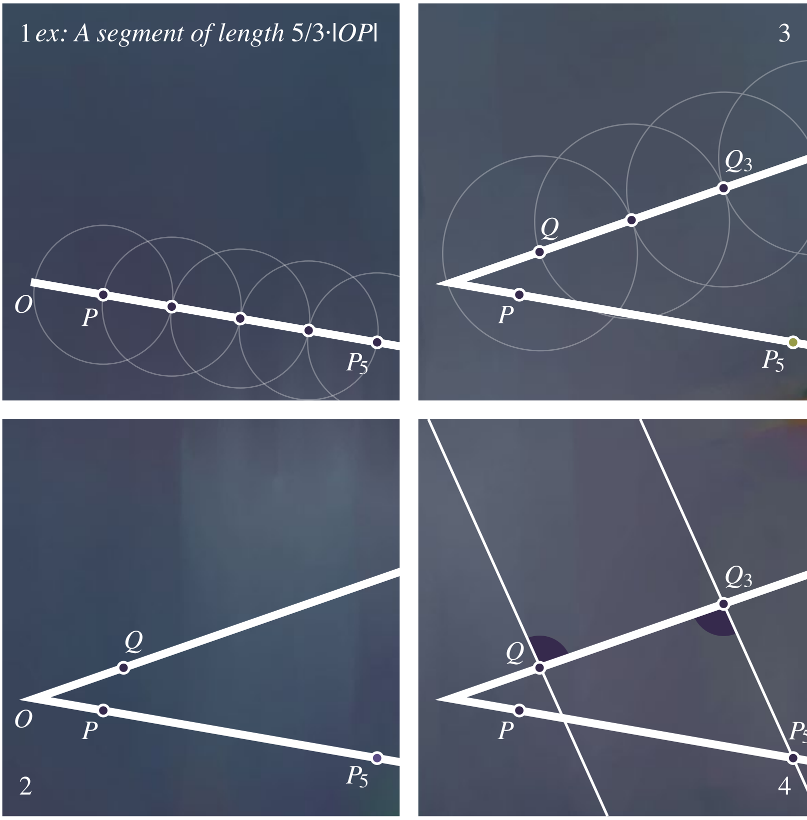


3 Construct the ray $\overrightarrow{QP}$.

4 This ray and ℓ form two angles, one on each side of $\overrightarrow{QP}$. Choose one of these two angles and call it θ.

5 Transfer this angle to another congruent angle θ' which comes off of the ray $\overrightarrow{PQ}$. There are two such angles, one on each side of the ray, but for the purposes of this construction, we want the one on the opposite side of $\overrightarrow{PQ}$ from θ.

6 Now $\overrightarrow{PQ}$ is one of the rays defining θ'. Extend the other ray to the line containing it: call this line ℓ'. By the Alternate Interior Angle Theorem, ℓ' is parallel to ℓ.



A rational multiple of a segment

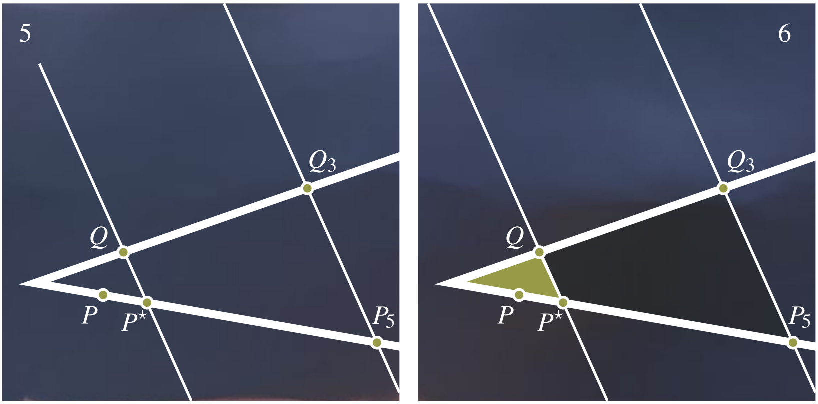
Given a segment OP, we can construct a segment whose length is any rational multiple m/n of $|OP|$.

1 Along $OP\rightarrow$, lay down m congruent copies of OP, end-to-end, to create a segment of length $m|OP|$. Label the endpoint of this segment as P_m.

2 Draw another ray with endpoint O (other than $OP \rightarrow$ or $OP \rightarrow^{op}$), and label a point on it Q.

3 Along $OQ\rightarrow$, lay down n congruent copies of OQ, end-to-end, to create a segment of length $n|OQ|$. Label the endpoint of this segment as Q_n.

4 Draw $\leftarrow P_mQ_n \rightarrow$ and construct the line through Q that is parallel to $\leftarrow P_mQ_n\rightarrow$.



5 It intersects $OP\rightarrow$. Label the intersection as $P^\star$.

6 I claim that $OP^\star$ is the segment we want: that

$$|OP|^\star = m/n \cdot |OP|.$$

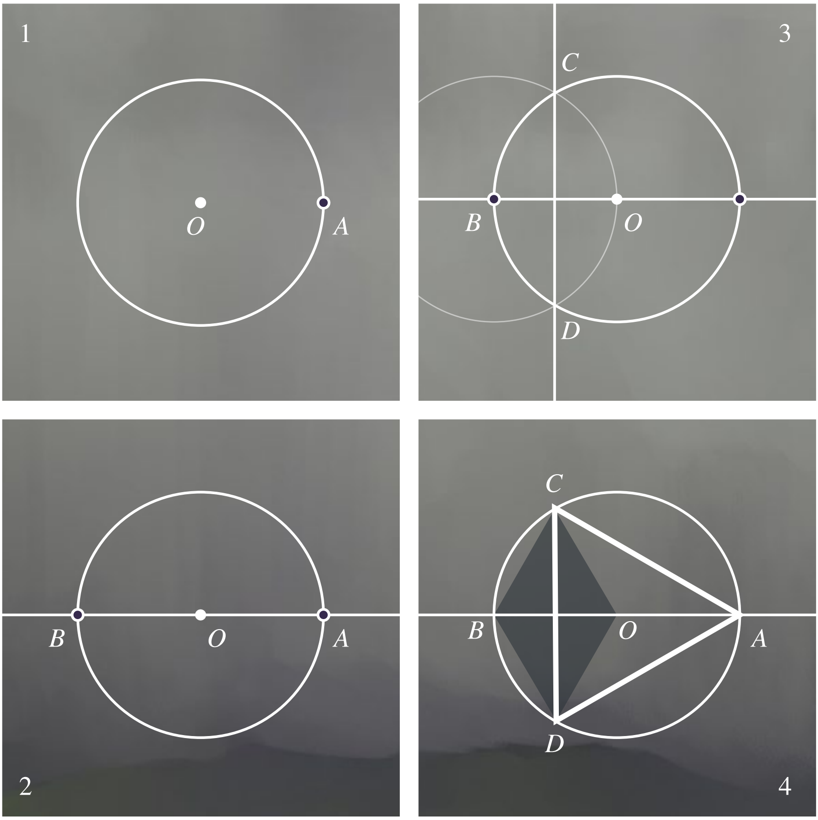
To see why, observe that O, $P^\star$, and P_n are all parallel projections from O, Q, and Q_m, respectively. Therefore,

$$\frac{|OP^\star|}{|OP_m|} = \frac{|OQ|}{|OQ_n|}$$

$$\frac{|OP^\star|}{m \cdot |OP|} = \frac{1}{n}$$

$$|OP^\star| = \frac{m}{n}|OP|.$$

To round out this lesson I would like to look at one of the central questions in the classical theory of constructions: given a circle, is it possible to construct a regular n-gon inscribed in it? This question has now been answered: it turns out that the answer is yes for some values of n, but no for others. In fact, a regular n-gon can be constructed if and only if n is a power of 2, or a product of a power of 2 and distinct Fermat primes (a Fermat prime is a prime of the form $2^{2^n}+1$, and the only known Fermat primes are 3, 5, 17, 257, and 65537). A proof of this result falls outside the scope of this book, but I would like to look at a few of the small values of n where the construction *is* possible. In all cases, the key is to construct a central angle at O which measures $2\pi/n$.



An equilateral triangle that is inscribed in a given circle

In this case, we need to construct a central angle of $2\pi/3$, and this can be done by constructing the supplementary angle of $\pi/3$.

1 Given a circle $\mathcal{C}$ with center O, mark a point A on it.

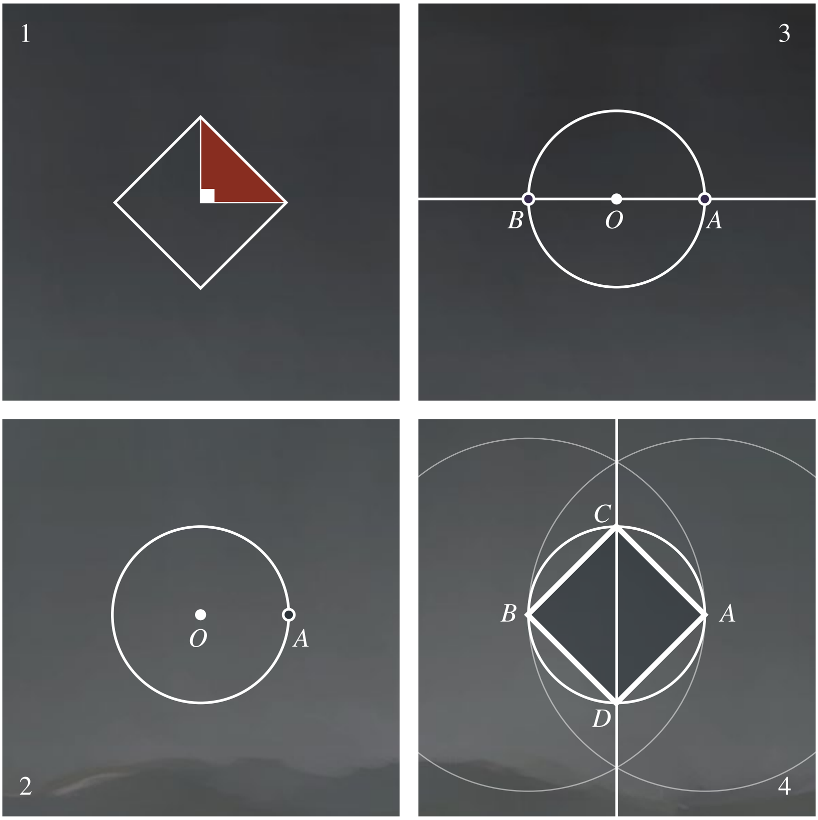
2 Draw the diameter through A, and mark the other endpoint of it as B.

3 Construct the perpendicular bisector to OB. Mark the intersections of that line with $\mathcal{C}$ as C and D.

4 The triangles $\triangle BOC$ and $\triangle BOD$ are equilateral, so

$$(\angle BOC) = (\angle BOD) = \pi/3$$

and so the two supplementary angles $\angle AOC$ and $\angle AOD$ each measure $2\pi/3$. Construct the segments AC and AD to complete the equilateral triangle $\triangle ACD$.



A square inscribed in a given circle

1 This is even easier, since the central angle needs to measure $\pi/2$– a right angle.

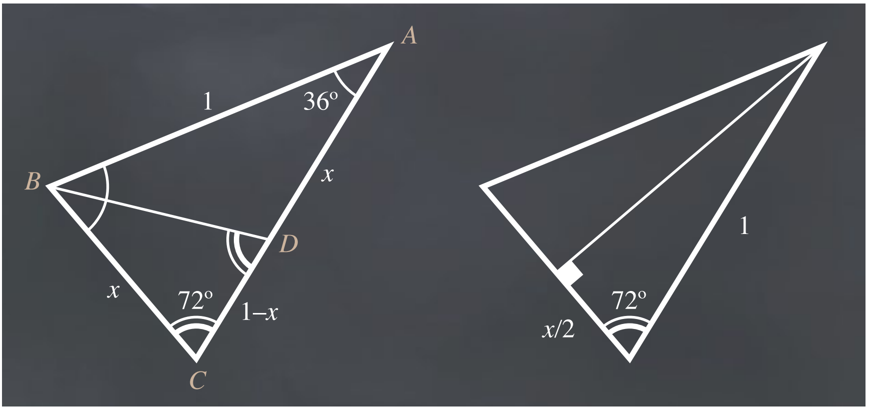
2 Given a circle $\mathcal{C}$ with center O, mark a point A on it.

3 Draw the diameter through A and mark the other endpoint as B.

4 Construct the perpendicular bisector to AB and mark the intersections with $\mathcal{C}$ as C and D. The four points A, B, C, and D are the vertices of the square. Just connect the dots to get the square itself.

A regular pentagon inscribed in a given circle

This one is considerably trickier. The central angle we are going to need is $2\pi/5$ (which is 72°), an angle that you see a lot less frequently than the $2\pi/3$ and the $\pi/2$ of the previous constructions. Before diving into the construction, then, let's take a little time to investigate the geometry of an angle measuring $2\pi/5$. Let me show you a configuration of isosceles triangles that answers a lot of questions.



In this illustration $AB \simeq AC$ and $BC \simeq BD$. Since $\triangle ABC \sim \triangle BCD$, we have a way to solve for x,

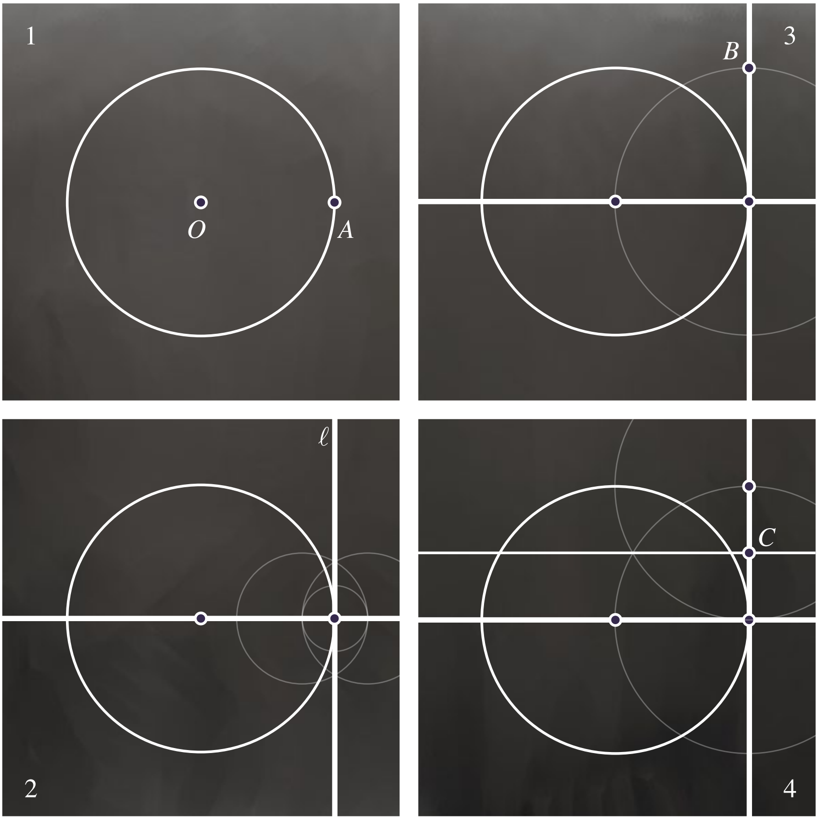
$$\frac{1-x}{x} = \frac{x}{1} \implies 1-x = x^2 \implies x^2+x-1=0$$

and with the quadratic formula, $x = (-1 \pm \sqrt{5})/2$. Of these solutions, x has to be the positive value since it represents a distance. The line from A to the midpoint of BC divides $\triangle ABC$ into two right triangles, and from them we can read off that

$$\cos(2\pi/5) = \frac{x/2}{1} = \frac{-1+\sqrt{5}}{4}.$$

This cosine value is the key to the construction of the regular pentagon.

[note: I am going with this construction because it seems pretty intuitive, but it is not the most efficient construction. Also, I am going to inscribe this pentagon in a circle of radius one to make the calculations a little easier– the same construction works in a circle of any radius though.]



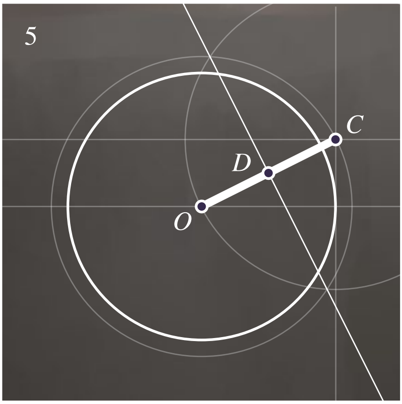
1 Given a circle $\mathcal{C}$ with center O and radius one. Mark a point A on $\mathcal{C}$.

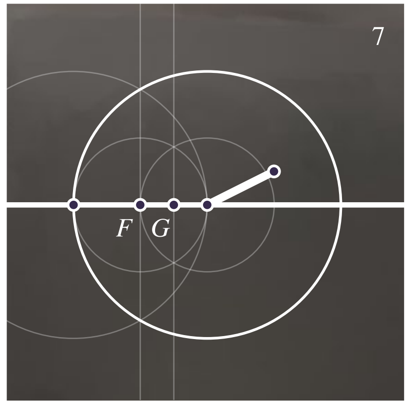
Objective I. Construct a segment of length $\sqrt{5}/4$.

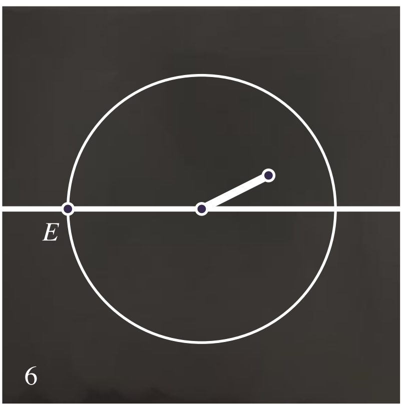
2 Construct the line which passes through A and is perpendicular to $\overleftrightarrow{OA}$. Call this line ℓ.

3 Use the compass to mark a point B on ℓ that is a distance $|OA|$ from A.

4 Construct the midpoint of AB, and call that point C.







5 Draw the segment OC. Note that by the Pythagorean Theorem,

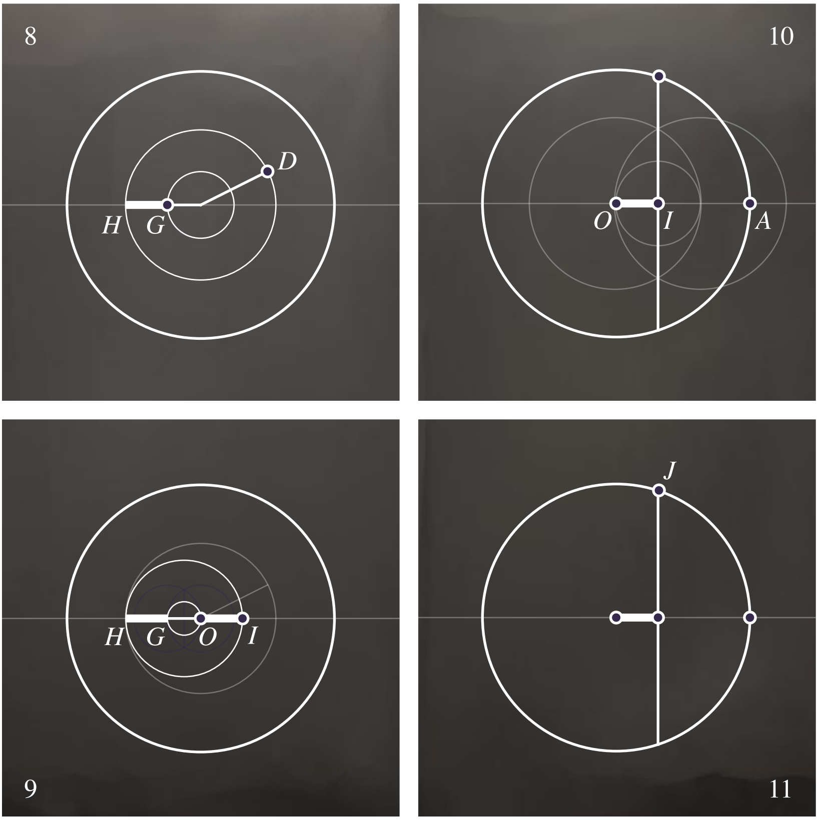
$$\begin{aligned}|OC| &= \sqrt{|OA|^2+|AC|^2}\\ &= \sqrt{1+(1/2)^2}\\ &= \sqrt{5}/2.\end{aligned}$$

Locate the midpoint of OC (which is a distance $\sqrt{5}/4$ from O). Call this point D.

Objective II. Construct a segment of length $1/4$.

6 Extend OA until it reaches the other side of $\mathcal{C}$ (the other endpoint of the diameter). Label this point E.

7 Find the midpoint F of OE, and then find the midpoint G of OF. Then $|OE|=1$, $|OF|=1/2$ and $|OG|=1/4$.



Objective III. Construct a segment of length $(-1+\sqrt{5})/4$.

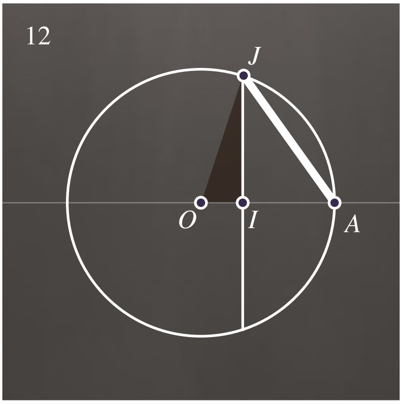
8 Draw the circle centered at O that passes through D. Mark its intersection with OE as H. Then GH is a segment whose length is $(-1+\sqrt{5})/4$.

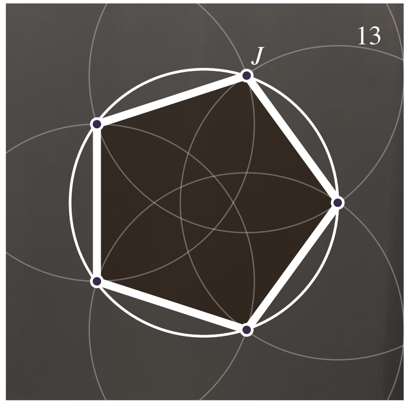
9 Use segment transfer to place a congruent copy of GH along the ray $OA\rightarrow$, with one endpoint at O. Label the other endpoint I.

Objective IV. Mark a vertex of the pentagon.

10 We will use A as one vertex of the pentagon. For the next, construct the line perpendicular to OA which passes through I.

11 Mark one of the intersections of this perpendicular with $\mathcal{C}$ as J.





12 Now look at $\angle O$ in the right triangle $\triangle OIJ$

$$\cos(\angle O) = \frac{|OI|}{|OJ|} = \frac{(-1+\sqrt{5})/4}{1}.$$

According to our previous calculation, that means $(\angle OIJ) = 2\pi/5$.

Objective V. The pentagon itself.

13 Segment AJ is one of the sides of the pentagon. Now just transfer congruent copies of that segment around the circle to get the other four sides of the pentagon.

Exercises

1. Given a segment AB, construct a segment of length $(7/3)|AB|$.
2. In a given circle, construct a regular (i) octagon, (ii) dodecagon, (iii) decagon.
3. Given a circle $\mathcal{C}$ and a point A outside the circle, construct the lines through A that are tangent to $\mathcal{C}$.
4. Foreshadowing. (i) Given a triangle, construct the perpendicular bisectors to the three sides. (ii) Given a triangle, construct the three angle bisectors.

   We haven't discussed area yet, but if you are willing to do some things out of order, here are a few area-based constructions.

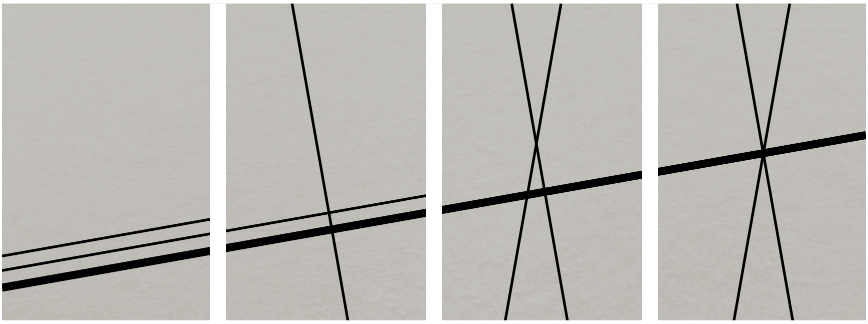
5. Given a square whose area is A, construct a square whose area is $2A$.
6. Given a rectangle, construct a square with the same area.
7. Given a triangle, construct a rectangle with the same area.

References

Famously, it is impossible to trisect an angle with compass and straight edge. The proof of this impossibility requires a little Galois Theory, but for the reader who has seen abstract algebra, is quite accessible. Proofs are often given in abstract algebra books– I like Durbin's approach in his *Modern Algebra* book [1](probably because it was the first one I saw).

[1] John R. Durbin. *Modern Algebra: An Introduction*. John Wiley and Sons, Inc., New York, 3rd edition, 1992.

19 CONCURRENCE I



Parallelism *Intersection* *Concurrence*

Start with three (or more) points. There is a small chance that those points all lie on the same line– that they are colinear. In all likelihood, though, they are not. And so, should we find a configuration of points that are consistently colinear, well, that could be a sign of something interesting. Likewise, with three (or more) lines, the greatest likelihood is that each pair of lines interect, but that none of the intersections coincide. It is unusual for two lines to be parallel, and it is unusual for three or more lines to intersect at the same point.

DEF: CONCURRENCE
When three (or more) lines all intersect at the same point, the lines are said to be *concurrent*. The intersection point is called the *point of concurrence*.

In this lesson we are going to look at a few (four) concurrences of lines associated with a triangle. Geometers have catalogued thousands of these concurrences, so this is just the tip of a very substantial iceberg. [1]

The circumcenter

In the last lesson, I gave the construction of the perpendicular bisector of a segment, but I am not sure that I ever properly defined it (oops). Let me fix that now.

DEF: PERPENDICULAR BISECTOR
The *perpendicular bisector* of a segment AB is the line which is perpendicular to AB and passes through its midpoint.

Our first concurrence deals with the perpendicular bisectors of the three sides of a triangle, but in order to properly understand that concurrence, we need another characterization of the points of the perpendicular bisector.

LEMMA

A point X is on the perpendicular bisector to AB if and only if

$$|AX| = |BX|.$$

Proof. There's not much to this proof. It is really just a simple application of some triangle congruence theorems. First, suppose that X is a point on the perpendicular bisector to AB and let M be the midpoint of AB. Then

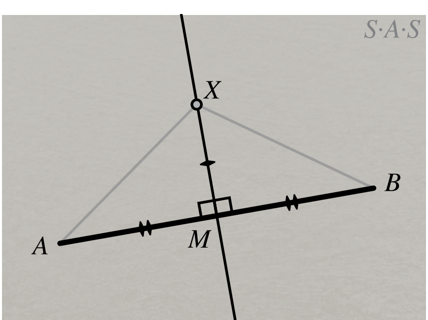
$$\begin{aligned} S &: AM \simeq BM \\ A &: \angle AMX \simeq \angle BMX \\ S &: MX = MX, \end{aligned}$$

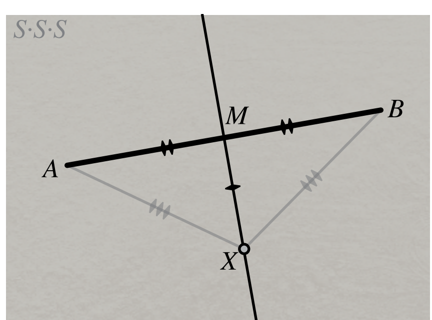
and so $\triangle AMX$ and $\triangle BMX$ are congruent. This means that $|AX| = |BX|$.

Conversely, suppose that $|AX| = |BX|$, and again let M be the midpoint of AB. Then

$$\begin{aligned} S &: AM \simeq BM \\ S &: MX = MX \\ S &: AX \simeq BX. \end{aligned}$$

and so $\triangle AMX$ and $\triangle BMX$ are congruent. In particular, this means that $\angle AMX \simeq \angle BMX$. Those two angles are supplements, though, and so they must be right angles. Hence X is on the line through M that forms a right angle with AB– it is on the perpendicular bisector. □

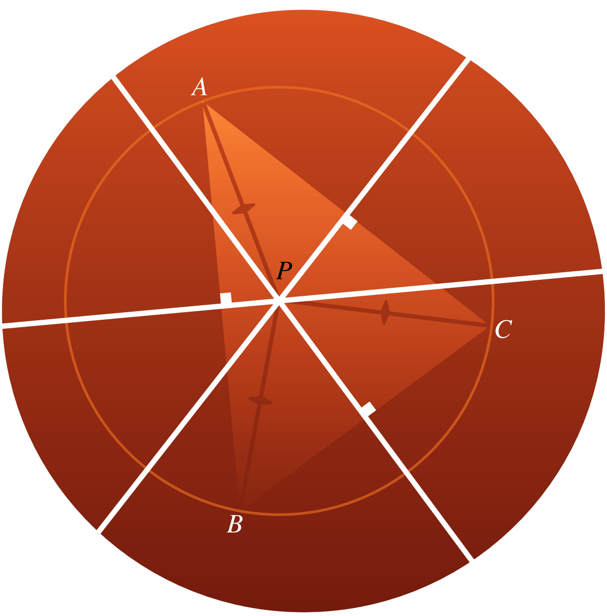




Now we are ready for the first concurrence.

THE CIRCUMCENTER
The perpendicular bisectors to the three sides of a triangle $\triangle ABC$ intersect at a single point. This point of concurrence is called the *circumcenter* of the triangle.

Proof. The first thing to notice is that no two sides of the triangle can be parallel. Therefore, none of the perpendicular bisectors can be parallel–they all intersect each other. Let P be the intersection point of the perpendicular bisectors to AB and BC. Since P is on the perpendicular bisector to AB, $|PA| = |PB|$. Since P is on the perpendicular bisector to BC, $|PB| = |PC|$. Therefore, $|PA| = |PC|$, and so P is on the perpendicular bisector to AC. □

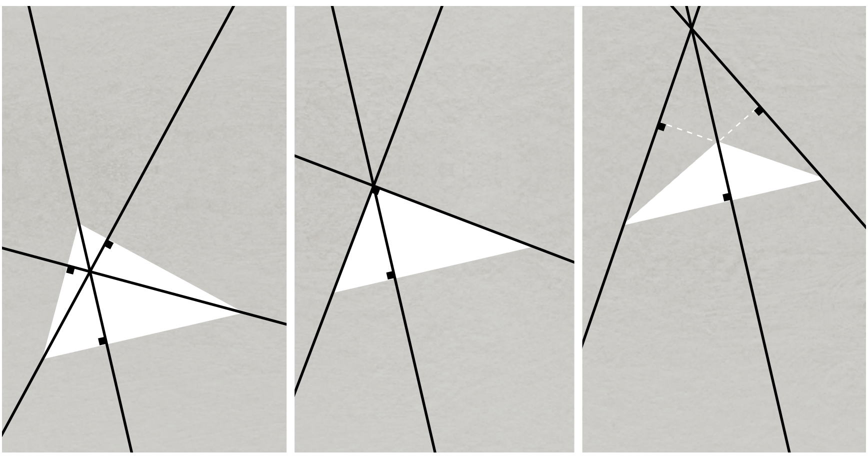


An important side note: P is equidistant from A, B and C. That means that there is a circle centered at P which passes through A, B, and C. This circle is called the *circumcircle* of $\triangle ABC$. In fact, it is the only circle which passes through all three of A, B, and C (which sounds like a good exercise).

The orthocenter

Most people will be familiar with the altitudes of a triangle from area calculations in elementary geometry. Properly defined,

DEF: ALTITUDE
An *altitude* of a triangle is a line which passes through a vertex and is perpendicular to the opposite side.



Altitudes for an acute, right, and obtuse triangle.

You should notice that an altitude of a triangle does not have to pass through the interior of the triangle at all. If the triangle is acute then all three altitudes will cross the triangle interior, but if the triangle is right, two of the altitudes will lie along the legs, and if the triangle is obtuse, two of the altitudes will only touch the triangle at their respective vertices. In any case, though, the altitude from the largest angle *will* cross through the interior of the triangle.

THE ORTHOCENTER
The three altitudes of a triangle $\triangle ABC$ intersect at a single point. This point of concurrence is called the *orthocenter* of the triangle.

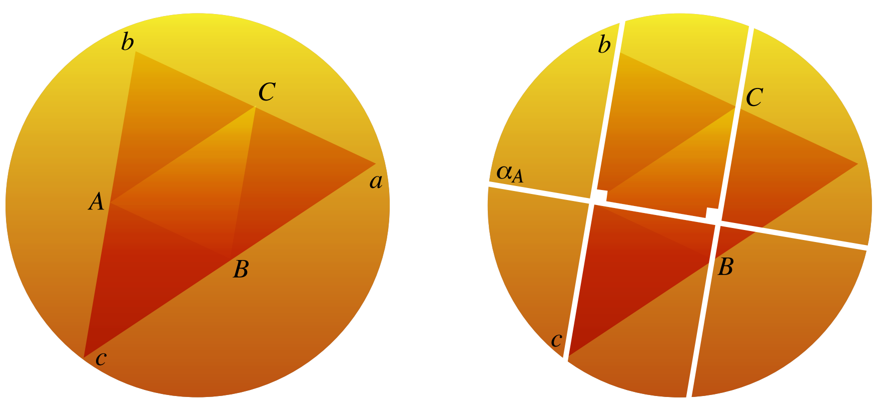
Proof. The key to this proof is that the altitudes of $\triangle ABC$ also serve as the perpendicular bisectors of another (larger) triangle. That takes us back to what we have just shown– that the perpendicular bisectors of a triangle are concurrent. First, we have to build that larger triangle. Draw three lines

ℓ_1 which passes through A and is parallel to BC,
ℓ_2 which passes through B and is parallel to AC,
ℓ_3 which passes through C and is parallel to AB.

Each pair of those lines intersect (they cannot be parallel since the sides of $\triangle ABC$ are not parallel), for a total of three intersections

$$\ell_1 \cap \ell_2 = c \quad \ell_2 \cap \ell_3 = a \quad \ell_3 \cap \ell_1 = b.$$

The triangle $\triangle abc$ is the "larger triangle". Now we need to show that an altitude of $\triangle ABC$ is a perpendicular bisector of $\triangle abc$. The argument is the same for each altitude (other than letter shuffling), so let's just focus on the altitude through A: call it α_A. I claim that α_A is the perpendicular bisector to bc. There are, of course, two conditions to show: (1) that $\alpha_A \perp bc$ and (2) that their intersection, A, is the midpoint of bc.



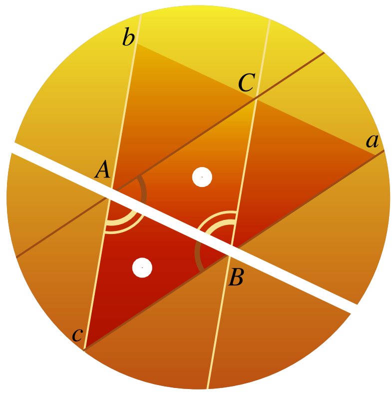
(1) The first is easy thanks to the simple interplay between parallel and perpendicular lines in Euclidean geometry.

$$bc \parallel BC \quad \& \quad BC \perp \alpha_A \quad \Longrightarrow \quad bc \perp \alpha_A.$$

(2) To get at the second, we are going to have to leverage some of the congruent triangles that we have created.

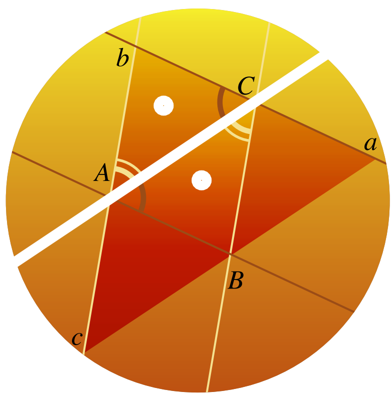
$$\begin{aligned}
A:&\quad AC \parallel ac \implies \angle cBA \simeq \angle BAC \\
S:&\quad AB = AB \\
A:&\quad BC \parallel bc \implies \angle cAB \simeq \angle ABC
\end{aligned}$$

$$\therefore \quad \triangle ABc \simeq \triangle BAC.$$

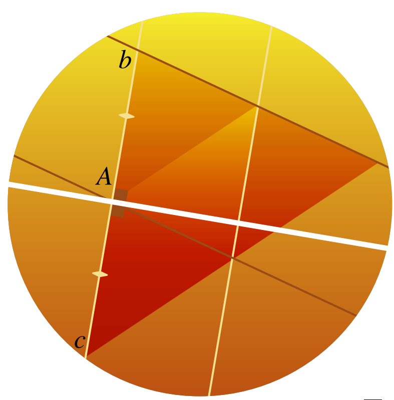


$$\begin{aligned}
A:&\quad AB \parallel ab \implies \angle BAC \simeq \angle bCA \\
S:&\quad AC = AC \\
A:&\quad BC \parallel bc \implies \angle BCA \simeq \angle bAC
\end{aligned}$$

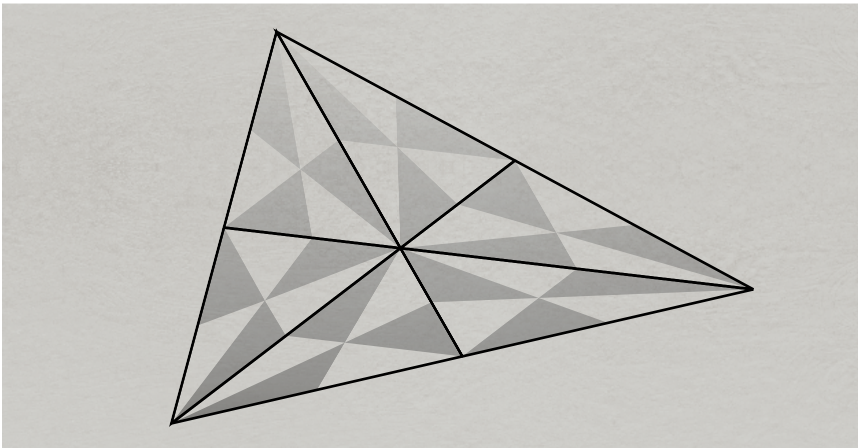
$$\therefore \quad \triangle ABC \simeq \triangle CbA.$$



Therefore $Ac \simeq BC \simeq Ab$, placing A at the midpoint of bc and making α_A the perpendicular bisector to bc. Likewise, the altitude through B is the perpendicular bisector to ac and the altitude through C is the perpendicular bisector to ab. As the three perpendicular bisectors of $\triangle abc$, these lines must intersect at a single point.



□



The three medians of a triangle

The centroid

MEDIAN
A *median* of a triangle is a line segment from a vertex to the midpoint of the opposite side.

THE CENTROID
The three medians of a triangle intersect at a single point. This point of concurrence is called the *centroid* of the triangle.

Proof. On $\triangle ABC$, label the midpoints of the three edges,

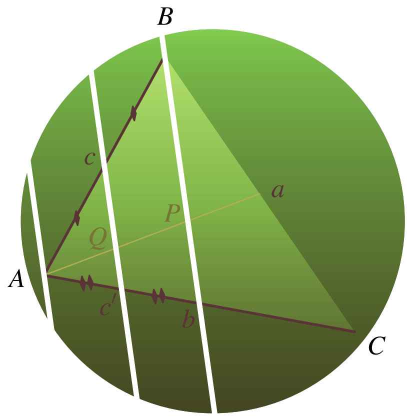
$$a, \text{ the midpoint of } BC,$$
$$b, \text{ the midpoint of } AC,$$
$$c, \text{ the midpoint of } AB,$$

so that Aa, Bb, and Cc are the medians. The key to this proof is that we can pin down the location of the intersection of any two medians– it will always be found two-thirds of the way down the median from the vertex. To understand why this is, we are going to have to look at a sequence of three parallel projections.

1. Label the intersection of Aa and Bb as P. Extend a line from c which is parallel to Bb. Label its intersection with Aa as Q, and its intersection with AC as c'. The first parallel projection, from AB to AC, associates the points

$$A \mapsto A \quad B \mapsto b \quad c \mapsto c'.$$

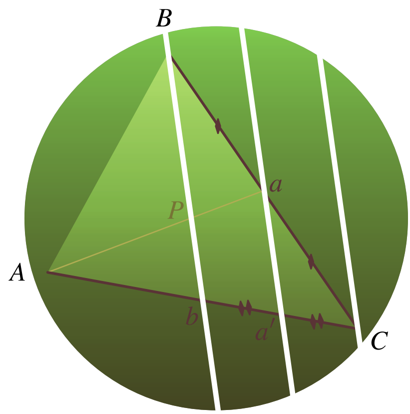
Since $Ac \simeq cB$, this means $Ac' \simeq c'b$.



2. Extend a line from a which is parallel to Bb. Label its intersection with AC as a'. The second parallel projection, from BC to AC, associates the points

$$C \mapsto C \quad B \mapsto b \quad a \mapsto a'.$$

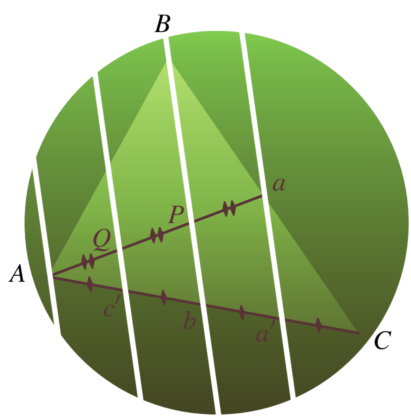
Since $Ca \simeq aB$, this means $Ca' \simeq a'b$.



3. Now b divides AC into two congruent segments, and a' and c' evenly subdivide them. In all, a', b, and c' split AC into four congruent segments. The third parallel projection is from AC back onto Aa:

$$A \mapsto A \quad c' \mapsto Q \quad b \mapsto P \quad a' \mapsto a.$$

Since $Ac' \simeq c'b \simeq ba'$, this means $AQ \simeq QP \simeq Pa$.



Therefore P, the intersection of Bb and Aa, will be found on Aa exactly two-thirds of the way down the median Aa from the vertex A. Now the letters in this argument are entirely arbitrary– with the right permutation of letters, we could show that any pair of medians will intersect at that two-thirds mark. Therefore, Cc will also intersect Aa at P, and so the three medians concur. □

Students who have taken calculus may already be familiar with the centroid (well, probably not my students, since I desperately avoid that section of the book, but students who have more conscientious and responsible teachers). In calculus, the centroid of a planar shape D can be thought of as its balancing point, and its coordinates can be calculated as

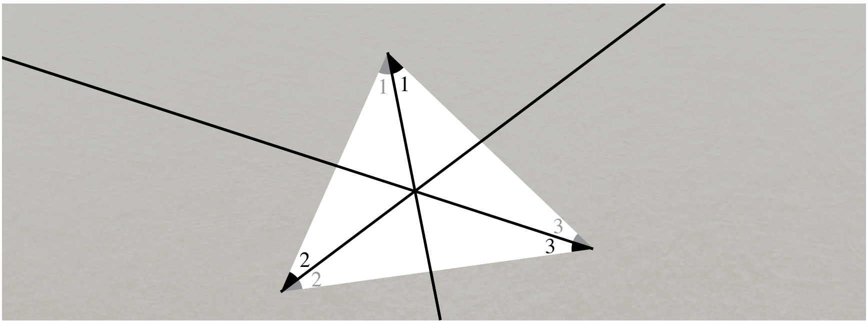
$$\frac{1}{\iint_D 1\,dxdy}\left(\iint_D x\,dxdy,\ \iint_D y\,dxdy\right).$$

It is worth noting (and an exercise for students who have done calculus) that in the case of triangles, the calculus and geometric definitions do coincide.

The incenter

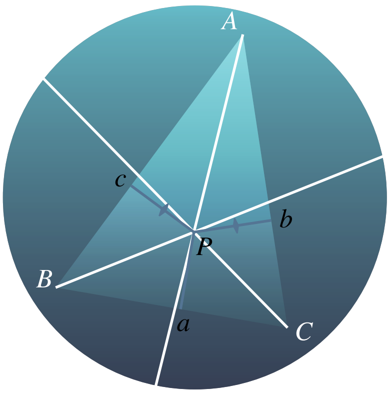
This lesson began with bisectors of the sides of a triangle. It seems fitting to end it with the bisectors of the interior angles of a triangle.

THE INCENTER
The bisectors of the three interior angles of a triangle intersect at a single point. This point of concurrence is called the *incenter* of the triangle.



Angle bisectors

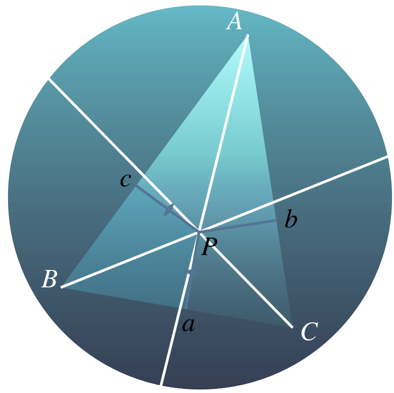
Proof. Take two of the angle bisectors, say the bisectors of $\angle A$ and $\angle B$, and label their intersection as P. We need to show that $CP\rightarrow$ bisects $\angle C$. The key to this proof is that P is actually equidistant from the three sides of $\triangle ABC$. From P, drop perpendiculars to each of the three sides of $\triangle ABC$. Label the feet of those perpendiculars: a on BC, b on AC, and c on AB.



Then

$$\begin{aligned} A:&\quad \angle PbA \simeq \angle PcA \\ A:&\quad \angle bAP \simeq \angle cAP \\ S:&\quad AP = AP \end{aligned}$$

so $\triangle AcP$ is congruent to $\triangle AbP$ and in particular $bP \simeq cP$.

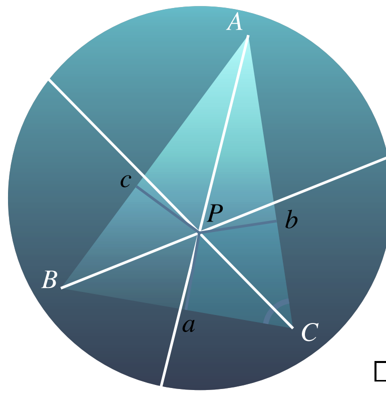


Again,

$$\begin{aligned} A:&\quad \angle PaB \simeq \angle PcB \\ A:&\quad \angle aBP \simeq \angle cBP \\ S:&\quad BP = BP \end{aligned}$$

and so $\triangle BaP$ is congruent to $\triangle BcP$ and in particular $cP \simeq aP$.

Now notice that the two right triangles $\triangle PaC$ and $\triangle PbC$ have congruent legs aP and bP and share the same hypotenuse PC. According to the H·L congruence theorem for right triangles, they have to be congruent. Thus, $\angle aCP \simeq \angle bCP$, and so $CP\rightarrow$ is the bisector of $\angle C$.



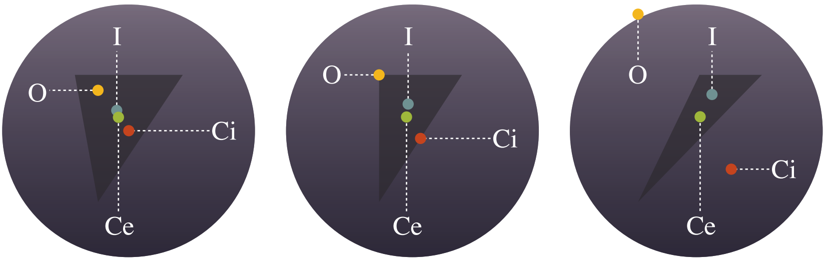
□

Notice that P is the same distance from each of the three feet a, b, and c. That means that there is a circle centered at P which is tangent to each of the three sides of the triangle. This is called the *inscribed circle*, or *incircle* of the triangle. It is discussed further in the exercises.

References

[1] Clark Kimberling. Encyclopedia of triangle centers - etc. distributed on World Wide Web. http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Triangle Centers



Ci	circumcenter	concurrence of perpendicular bisectors
O	orthocenter	concurrence of altitudes
Ce	centroid	concurrence of medians
I	incenter	concurrence of angle bisectors

Exercises

1. Using only compass and straight edge, construct the circumcenter, orthocenter, centroid, and incenter of a given triangle.

2. Using only compass and straight edge, construct the circumcircle and incircle of a given triangle.

3. Let A, B, and C be three non-colinear points. Show that the circumcircle to $\triangle ABC$ is the only circle passing through all three points A, B, and C.

4. Let A, B and C be three non-colinear points. Show that the incircle is the unique circle which is contained in $\triangle ABC$ and tangent to each of the three sides.

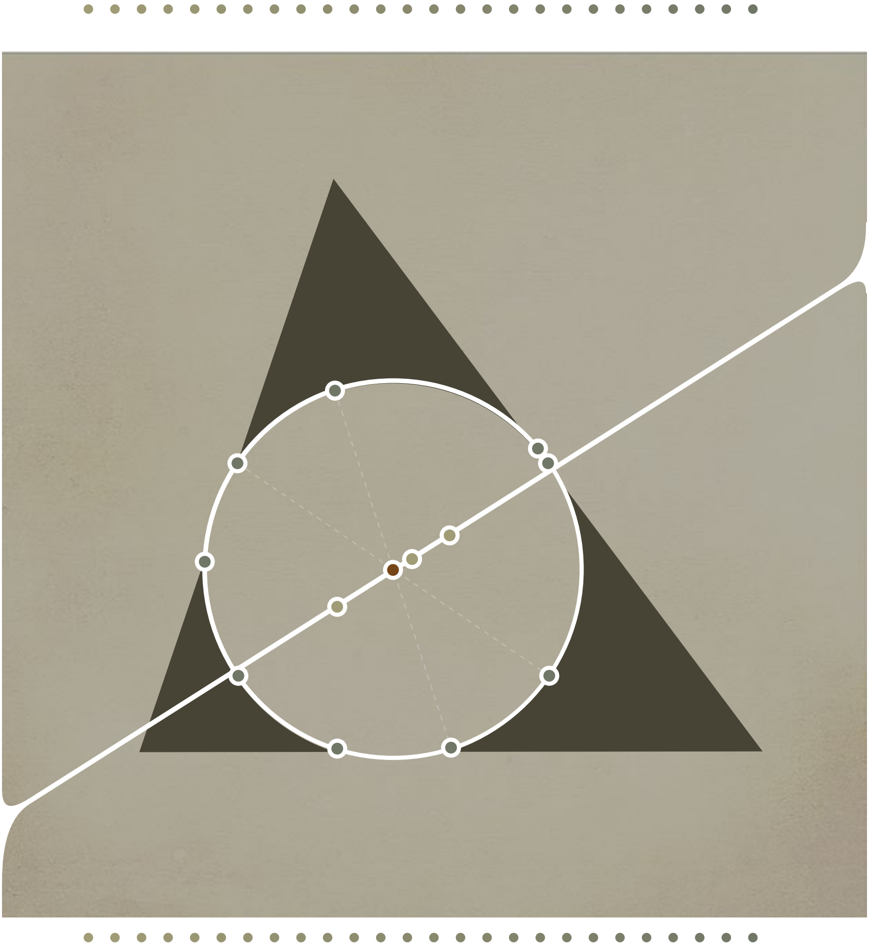
5. Show that the calculus definition and the geometry definition of the centroid of a triangle are the same.

6. Under what circumstances does the circumcenter of a triangle lie outside the triangle? What about the orthocenter?

7. Under what circumstances do the orthocenter and circumcenter coincide? What about the orthocenter and centroid? What about the circumcenter and centroid?

8. For any triangle $\triangle ABC$, there is an associated triangle called the orthic triangle whose three vertices are the feet of the altitudes of $\triangle ABC$. Prove that the orthocenter of $\triangle ABC$ is the incenter of its orthic triangle. [Hint: look for cyclic quadrilaterals and recall that the opposite angles of a cyclic quadrilateral are supplementary.]

9. Suppose that $\triangle ABC$ and $\triangle abc$ are similar triangles, with a scaling constant k, so that $|AB|/|ab| = k$. Let P be a center of $\triangle ABC$ (circumcenter, orthocenter, centroid, or incenter) and let p be the corresponding center of $\triangle abc$. (1) Show that $|AP|/|ap| = k$. (2) Let D denote the distance from P to AB and let d denote the distance from p to ab. Show that $D/d = k$.

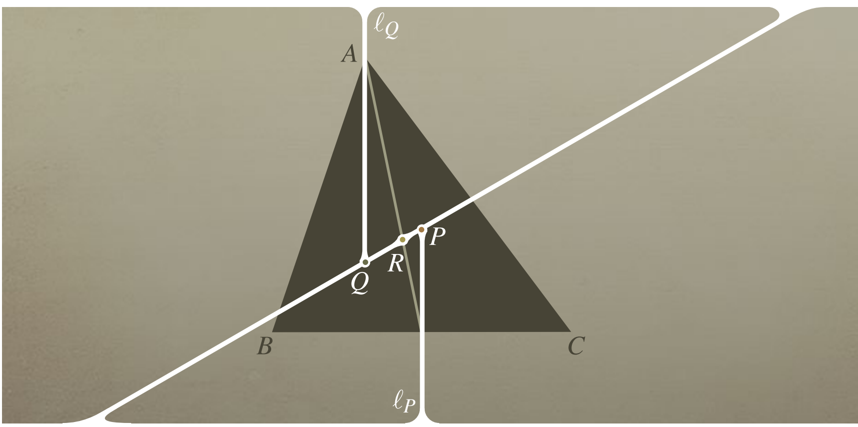


20 CONCURRENCE II

The Euler line

I wrapped up the last lesson with illustrations of three triangles and their centers, but I wonder if you noticed something in those illustrations? In each one, it certainly appears that the circumcenter, orthocenter, and centroid are colinear. Well, guess what– this is no coincidence.

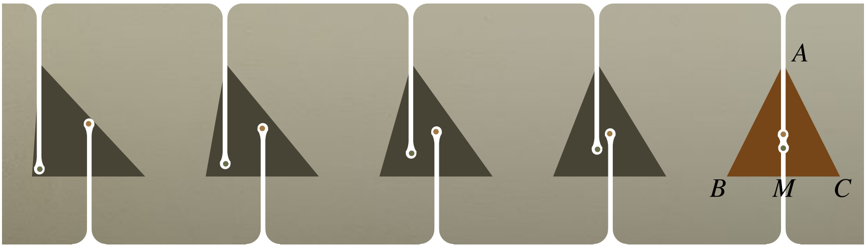
THM: THE EULER LINE
The circumcenter, orthocenter and centroid of a triangle are colinear, on a line called the *Euler line*.



Proof. First, the labels. On $\triangle ABC$, label

P: the circumcenter
Q: the orthocenter
R: the centroid
M: the midpoint of BC
ℓ_P: the perpendicular bisector to BC
ℓ_Q: the altitude through A
ℓ_R: the line containing the median AM

A dynamic sketch of all these points and lines will definitely give you a better sense of how they interact. Moving the vertices A, B, and C creates a rather intricate dance of P, Q and R. One of the most readily apparent features of this construction is that both ℓ_P and ℓ_Q are perpendicular to BC, and that means they cannot intersect unless they coincide. If you do have a sketch to play with, you will see that they *can* coincide.



Aligning an altitude and a perpendicular bisector.

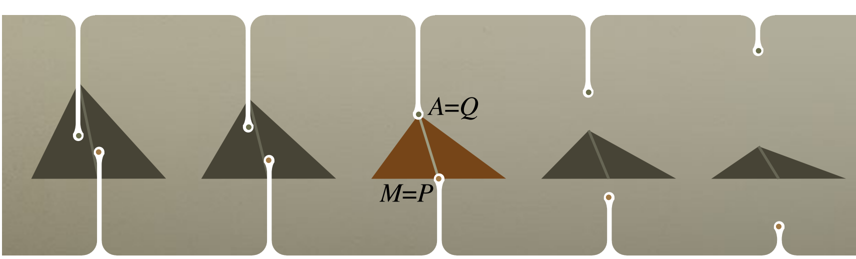
This is a good place to start the investigation.

$$\begin{aligned}
&\ell_P = \ell_Q \\
\iff\ &\ell_R \text{ intersects } BC \text{ at a right angle} \\
\iff\ &\triangle AMB \text{ is congruent to } \triangle AMC \\
\iff\ &AB \simeq AC
\end{aligned}$$

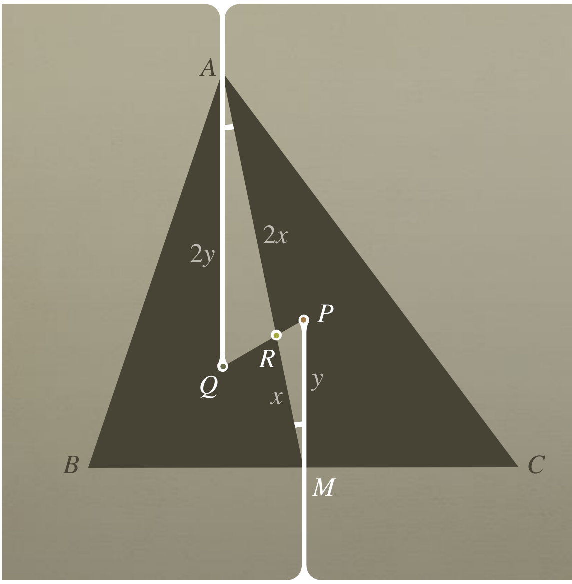
So in an isosceles triangle with congruent sides AB and AC, all three of P and Q and R will lie on the line $\ell_P = \ell_Q = \ell_R$. It is still possible to line up P, Q and R along the median AM without having ℓ_P, ℓ_Q and ℓ_R coincide. That's because ℓ_P intersects AM at M and ℓ_Q intersects AM at A, and it turns out that it is possible to place P at M and Q at A.

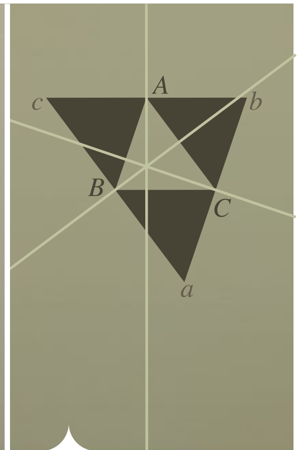
$$\begin{aligned}
&M \text{ is the circumcenter} \\
\iff\ &BC \text{ is a diameter of the circumcircle} \\
\iff\ &\angle A \text{ is a right angle (Thales' theorem)} \\
\iff\ &AB \text{ and } AC \text{ are both altitudes of } \triangle ABC \\
\iff\ &A \text{ is the orthocenter}
\end{aligned}$$

So if $\triangle ABC$ is a right triangle whose right angle is at vertex A, then again the median AM contains P, Q, and R.



Putting the circumcenter and orthocenter on a median.





The second S.
One triangle's altitudes are another triangle's perpendicular bisectors.

In all other scenarios, P and Q will *not* be found on the median, and this is where things get interesting. At the heart of this proof are two triangles, $\triangle AQR$ and $\triangle MPR$. We must show they are similar.

S: We saw in the last lesson that the centroid is located two thirds of the way down the median AM from A, so $|AR| = 2|MR|$.

A: $\angle QAR \simeq \angle PMR$, since they are alternate interior angles between the two parallel lines ℓ_P and ℓ_Q.

S: Q, the orthocenter of $\triangle ABC$, is also the circumcenter of another triangle $\triangle abc$. This triangle is similar to $\triangle ABC$, but twice as big. That means that the distance from Q, the circumcenter of $\triangle abc$ to side bc is double the distance from P, the circumcenter of $\triangle ABC$, to side BC (it was an exercise at the end of the last lesson to show that distances from centers are scaled proportionally by a similarity– if you skipped that exercise then, you should do it now, at least for this one case). In short, $|AQ| = 2|MP|$.

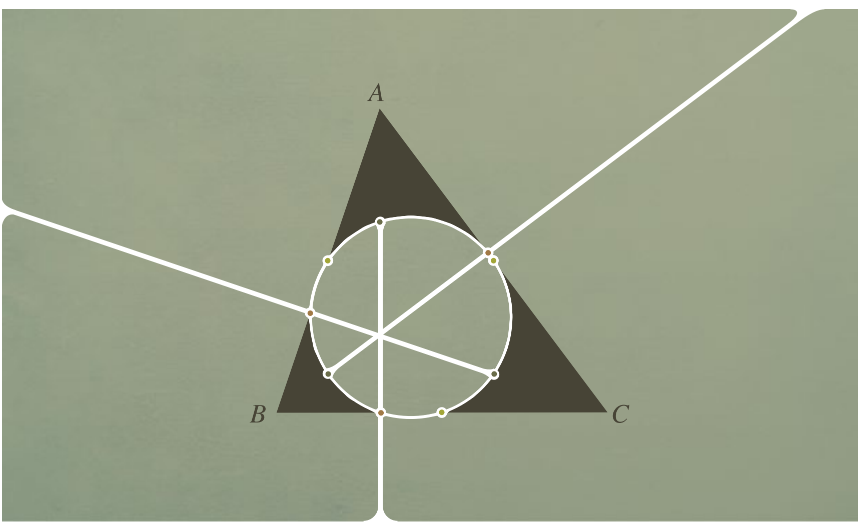
By S·A·S similarity, then, $\triangle AQR \sim \triangle MPR$. That means $\angle PRM$ is congruent to $\angle QRA$. The supplement of $\angle PRA$ is $\angle PRM$, so $\angle PRM$ must also be the supplement of $\angle QRA$. Therefore P, Q, and R are colinear. $\square$

The nine point circle

While only three points are needed to define a unique circle, the next result lists nine points associated with any triangle that are always on one circle. Six of the points were identified by Feuerbach (and for this reason the circle sometimes bears his name). Several more beyond the traditional nine have been found since. If you are interested in the development of this theorem, there is a brief history in *Geometry Revisited* by Coxeter and Greitzer [1].

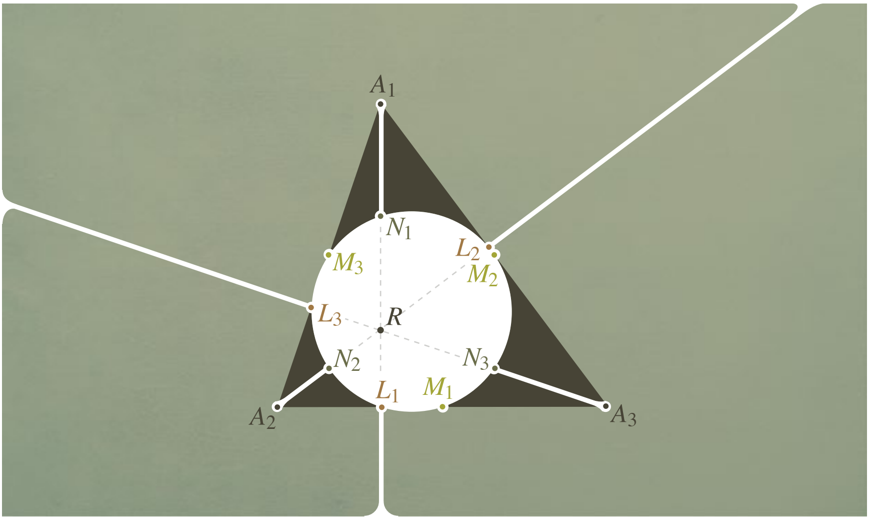
THM: THE NINE POINT CIRCLE

For any triangle, the following nine points all lie on the same circle: (1) the feet of the three altitudes, (2) the midpoints of the three sides, and (3) the midpoints of the three segments connecting the orthocenter to the each vertex. This circle is the *nine point circle* associated with that triangle.



This is a relatively long proof, and I would ask that you make sure you are aware of two key results that will play pivotal roles along the way.

1. Thales' Theorem: A triangle $\triangle ABC$ has a right angle at C if and only if C is on the circle with diameter AB.
2. The diagonals of a parallelogram bisect one another.



Proof. Given the triangle $\triangle A_1A_2A_3$ with orthocenter R, label the following nine points:

> L_i, the foot of the altitude which passes through A_i,
> M_i, the midpoint of the side that is opposite A_i,
> N_i, the midpoint of the segment A_iR.

The proof that I give here is based upon a key fact that is *not* mentioned in the statement of the theorem– that the segments M_iN_i are diameters of the nine point circle. We will take $\mathcal{C}$, the circle with diameter M_1N_1 and show that the remaining seven points are all on it. Allow me a moment to outline the strategy. First, we will show that the four angles

$$\angle M_1M_2N_1 \quad \angle M_1N_2N_1 \quad \angle M_1M_3N_1 \quad \angle M_1N_3N_1$$

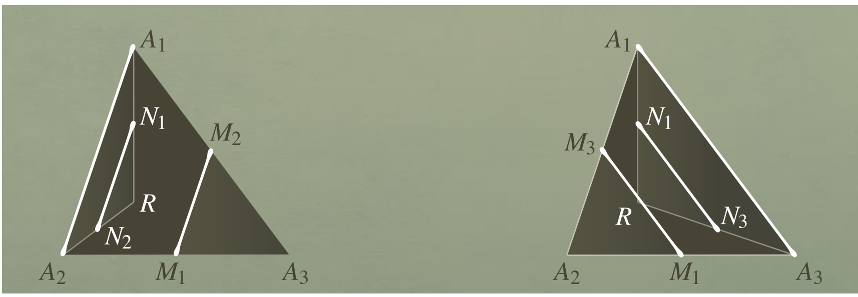
are right angles. By Thales' Theorem, that will place each of the points M_2, M_3, N_2, and N_3 on $\mathcal{C}$. Second, we will show that M_2N_2 and M_3N_3 are in fact diameters of $\mathcal{C}$. Third and finally, we will show that each $\angle M_iL_iN_i$ is a right angle, thereby placing the L_i on $\mathcal{C}$.

Lines that are parallel.
We need to prove several sets of lines are parallel to one another. The key in each case is S·A·S triangle similarity, and the argument for that similarity is the same each time. Let me just show you with the first one, and then I will leave out the details on all that follow.

Observe in triangles $\triangle A_3M_1M_2$ and $\triangle A_3A_2A_1$ that

$$|A_3M_2| = \tfrac{1}{2}|A_3A_1| \quad \angle A_3 = \angle A_3 \quad |A_3M_1| = \tfrac{1}{2}|A_3A_2|.$$

By the S·A·S similarity theorem, then, they are similar. In particular, the corresponding angles $\angle M_2$ and $\angle A_1$ in those triangles are congruent. According to the Alternate Interior Angle Theorem, M_1M_2 and A_1A_2 must be parallel. Let's employ that same argument many more times.

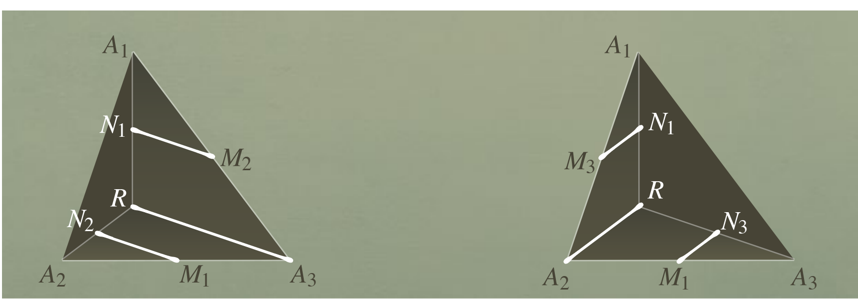


$\triangle A_3M_1M_2 \sim \triangle A_3A_2A_1$
$\implies M_1M_2 \parallel A_1A_2$

$\triangle RN_1N_2 \sim \triangle RA_1A_2$
$\implies N_1N_2 \parallel A_1A_2$

$\triangle A_2M_1M_3 \sim \triangle A_2A_3A_1$
$\implies M_1M_3 \parallel A_1A_3$

$\triangle RN_1N_3 \sim \triangle RA_1A_3$
$\implies N_1N_3 \parallel A_1A_3$

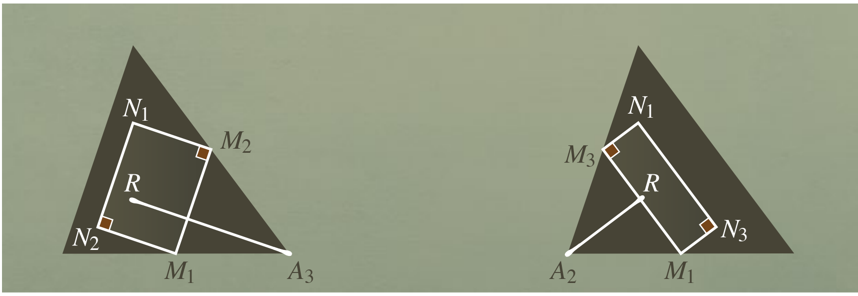


$\triangle A_1N_1M_2 \sim \triangle A_1RA_3$
$\implies N_1M_2 \parallel A_3R$

$\triangle A_2M_1N_2 \sim \triangle A_2A_3R$
$\implies M_1N_2 \parallel A_3R$

$\triangle A_1M_3N_1 \sim \triangle A_1A_2R$
$\implies M_3N_1 \parallel A_2R$

$\triangle A_3M_1N_3 \sim \triangle A_3A_2R$
$\implies M_1N_3 \parallel A_2R$



Angles that are right.

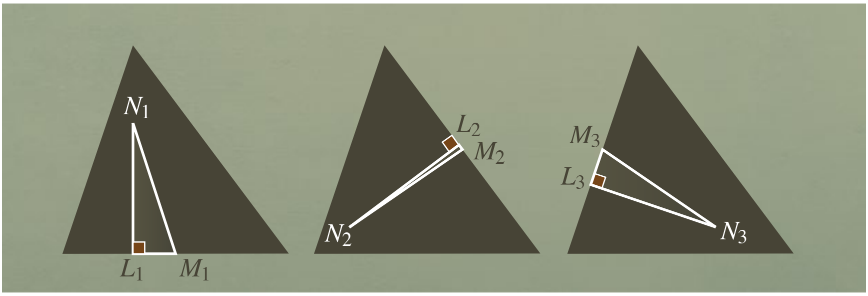
Now A_3R is a portion of the altitude perpendicular to A_1A_2. That means the first set of parallel lines are all perpendicular to the second set of parallel lines. Therefore M_1M_2 and M_2N_1 are perpendicular, so $\angle M_1M_2N_1$ is a right angle; and N_1N_2 and N_2M_1 are perpendicular, so $\angle M_1N_2N_1$ is a right angle. By Thales' Theorem, both M_2 and N_2 are on $\mathcal{C}$.

Similarly, segment A_2R is perpendicular to A_1A_3 (an altitude and a base), so M_1M_3 and M_3N_1 are perpendicular, and so $\angle M_1M_3N_1$ is a right angle. Likewise, N_1N_3 and N_3M_1 are perpendicular, so $\angle M_1N_3N_1$ is a right angle. Again Thales' Theorem tells us that M_3 and N_3 are on $\mathcal{C}$.

Segments that are diameters.
We have all the M's and N's placed on $\mathcal{C}$ now, but we aren't done with them just yet. Remeber that M_1N_1 is a diameter of $\mathcal{C}$. From that, it is just a quick hop to show that L_1 is also on $\mathcal{C}$. It would be nice to do the same for L_2 and L_3, but in order to do that we will have to know that M_2N_2 and M_3N_3 are also diameters. Based upon our work above,

$$M_1M_2 \parallel N_1N_2 \quad \& \quad M_1N_2 \parallel M_2N_1$$

That makes $\square M_1M_2N_1N_2$ a parallelogram (in fact it is a rectangle). Its two diagonals, M_1N_1 and M_2N_2 must bisect each other. In other words, M_2N_2 crosses M_1N_1 at its midpoint. Well, the midpoint of M_1N_1 is the center of $\mathcal{C}$. That means that M_2N_2 passes through the center of $\mathcal{C}$, and that makes it a diameter. The same argument works for M_3N_3. The parallelogram is $\square M_1M_3N_1N_3$ with bisecting diagonals M_1N_1 and M_3N_3.

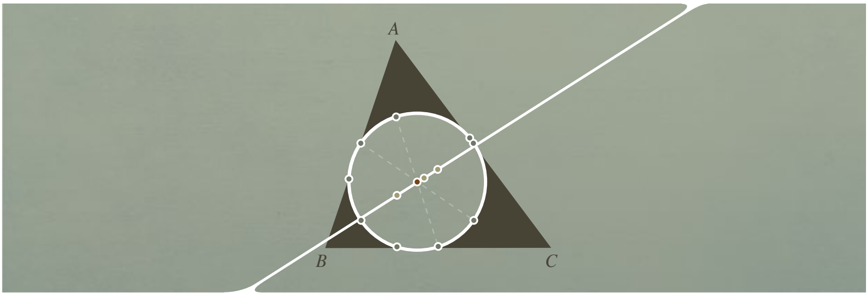


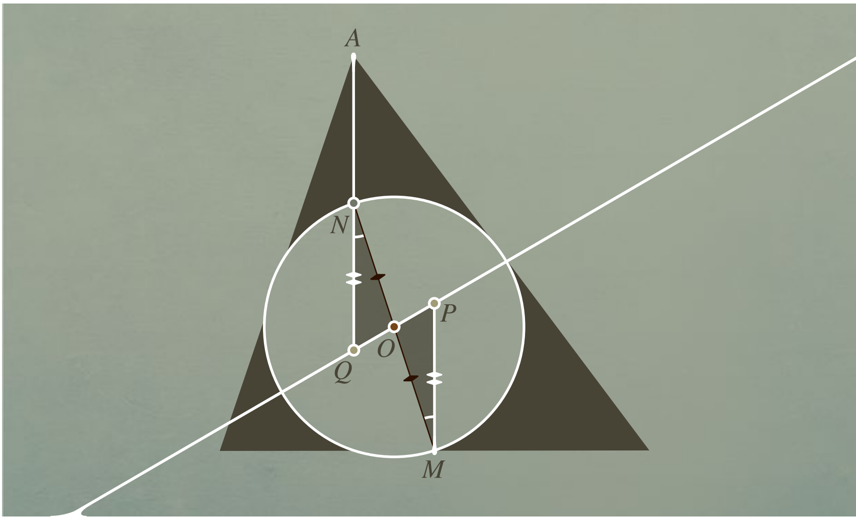
More angles that are right.
All three of M_1N_1, M_2N_2, and M_3N_3 are diameters of $\mathcal{C}$. All three of $\angle M_1L_1N_1$, $\angle M_2L_2N_2$ and $M_3L_3N_3$ are formed by the intersection of an altitude and a base, and so are right angles. Therefore, by Thales' Theorem, all three of L_1, L_2 and L_3 are on $\mathcal{C}$. $\square$

The center of the nine point circle

The third result of this lesson ties together the previous two.

THM
The center of the nine point circle is on the Euler line.





Proof. This proof nicely weaves together a lot of what we have developed over the last two lessons. On $\triangle ABC$, label the circumcenter P and the orthocenter Q. Then $\overleftrightarrow{PQ}$ is the Euler line. Label the center of the nine point circle as O. Our last proof hinged upon a diameter of the nine point circle. Let's recycle some of that– if M is the midpoint of BC and N is the midpoint of QA, then MN is a diameter of the nine point circle. Now this proof really boils down to a single triangle congruence– we need to show that $\triangle ONQ$ and $\triangle OMP$ are congruent.

S: $ON \simeq OM$. The center O of the nine point circle bisects the diameter MN.

A: $\angle M \simeq \angle N$. These are alternate interior angles between two parallel lines, the altitude and bisector perpendicular to BC.

S: $NQ \simeq MP$. In the Euler line proof we saw that $|AQ| = 2|MP|$. Well, $|NQ| = \frac{1}{2}|AQ|$, so $|NQ| = |MP|$.

By S·A·S, the triangles $\triangle ONQ$ and $\triangle OMP$ are congruent, and in particular $\angle QON \simeq \angle POM$. Since $\angle NOP$ is supplementary to $\angle POM$, it must also be supplementary to $\angle QON$. Therefore Q, O, and P are colinear, and so O is on the Euler line. $\square$

Exercises

1. Consider a triangle $\triangle ABC$. Let D and E be the feet of the altitudes on the sides AC and BC. Prove that there is a circle which passes through the points A, B, D, and E.

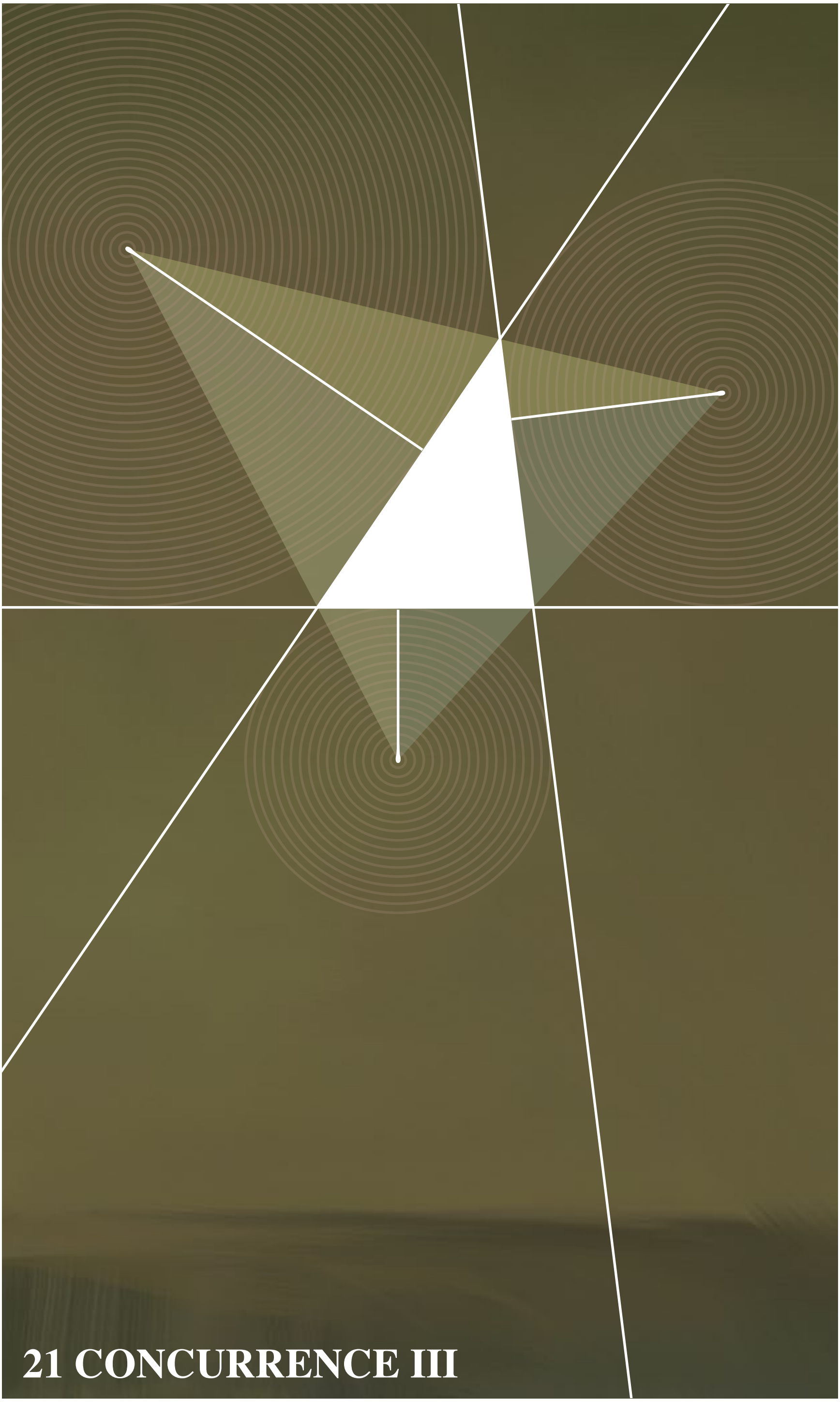
2. Under what conditions does the incenter lie on the Euler line?

3. Consider an isosceles triangle $\triangle ABC$ with $AB \simeq AC$. Let D be a point on the arc between B and C of the circumscribing circle. Show that DA bisects the angle $\angle BDC$.

4. Let P be a point on the circumcircle of triangle $\triangle ABC$. Let L be the foot of the perpendicular from P to AB, M be the foot of the perpendicular from P to AC, and N be the foot of the perpendicular from P to BC. Show that L, M, and N are collinear. This line is called a *Simson line*. Hint: look for cyclic quadrilaterals and use the fact that opposite angles in a cyclic quadrilateral are congruent.

References

[1] H.S.M. Coxeter and Samuel L. Greitzer. *Geometry Revisited*. Random House, New York, 1st edition, 1967.

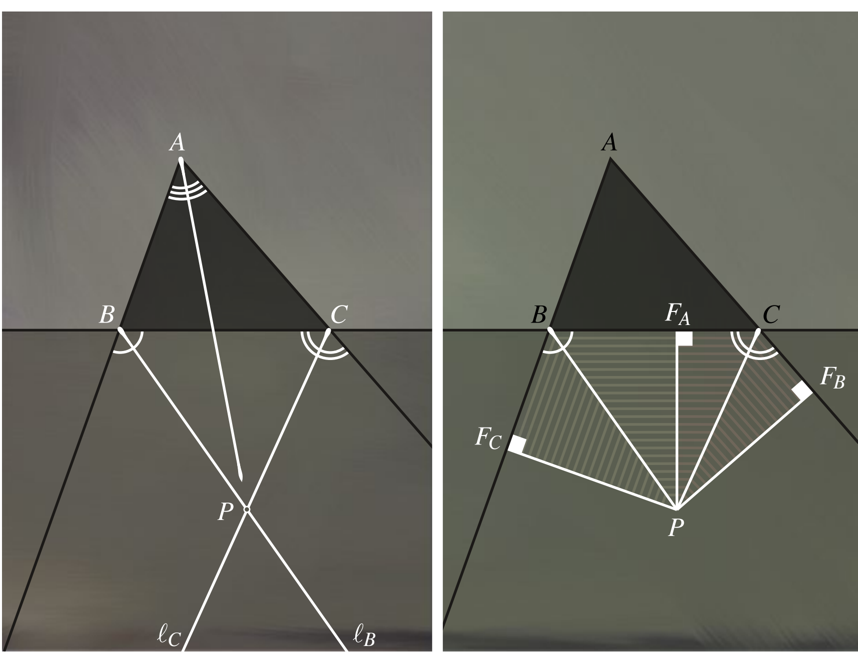


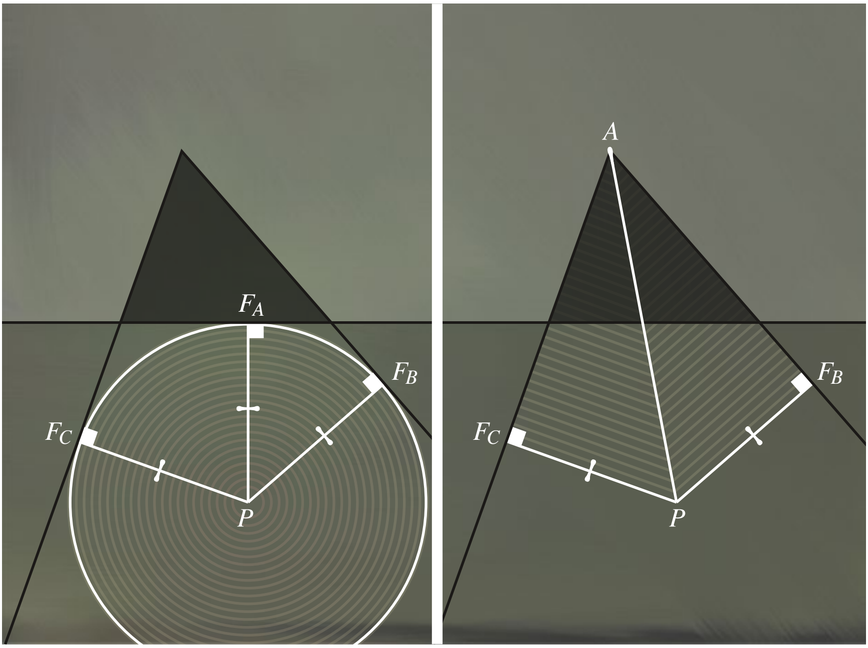
21 CONCURRENCE III

Excenters and excircles

In the first lesson on concurrence, we saw that the bisectors of the interior angles of a triangle concur at the incenter. If you did the exercise in the last lesson dealing with the orthic triangle then you may have noticed something else– that the sides of the original triangle are the bisectors of the exterior angles of the orthic triangle. I want to lead off this last lesson on concurrence with another result that connects interior and exterior angle bisectors.

THM: EXCENTERS
The exterior angle bisectors at two vertices of a triangle and the interior angle bisector at the third vertex of that triangle intersect at one point.





Proof. Let ℓ_B and ℓ_C be the lines bisecting the exterior angles at vertices B and C of $\triangle ABC$. They must intersect. Label the point of intersection as P. Now we need to show that the interior angle bisector at A must also cross through P, but we are going to have to label a few more points to get there. Let F_A, F_B, and F_C be the feet of the perpendiculars through P to each of the sides BC, AC, and AB, respectively. Then, by A·A·S,

$$\triangle PF_AC \simeq \triangle PF_BC \quad \triangle PF_AB \simeq \triangle PF_CB.$$

Therefore $PF_A \simeq PF_B \simeq PF_C$. Here you may notice a parallel with the previous discussion of the incenter– P, like the incenter, is equidistant from the lines containing the three sides of the triangle. By H·L right triangle congruence, $\triangle PF_CA \simeq \triangle PF_BA$. In particular, $\angle PAF_C \simeq \angle PAF_B$ and so P is on the bisector of angle A. □

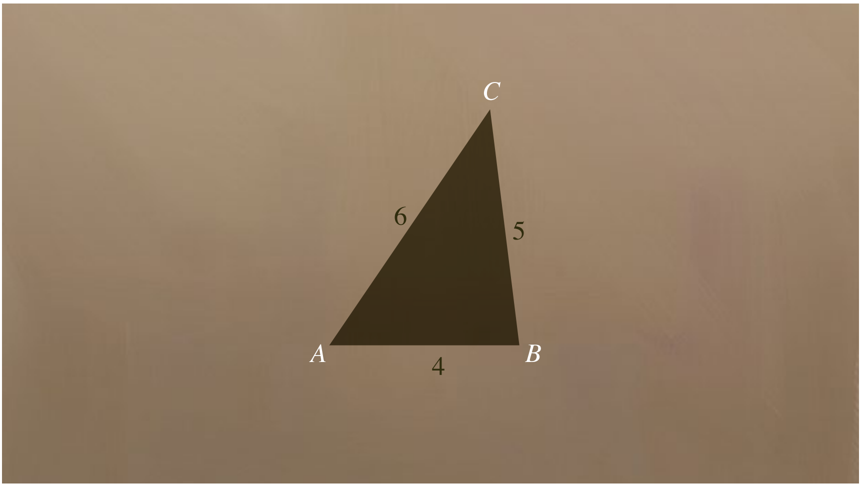
There are three such points of concurrence. They are called the *excenters* of the triangle. Since each is equidistant from the three lines containing the sides of the triangle, each is the center of a circle tangent to those three lines. Those circles are called the *excircles* of the triangle.

Ceva's Theorem

By now, you should have seen enough concurrence theorems and enough of their proofs to have some sense of how they work. Most of them ultimately turn on a few hidden triangles that are congruent or similar. Take, for example, the concurrence of the medians. The proof of that concurrence required a 2 : 1 ratio of triangles. What about other triples of segments that connect the vertices of a triangle to their respective opposite sides? What we need is a computation that will discriminate between triples of segments that do concur and triples of segments that do not.

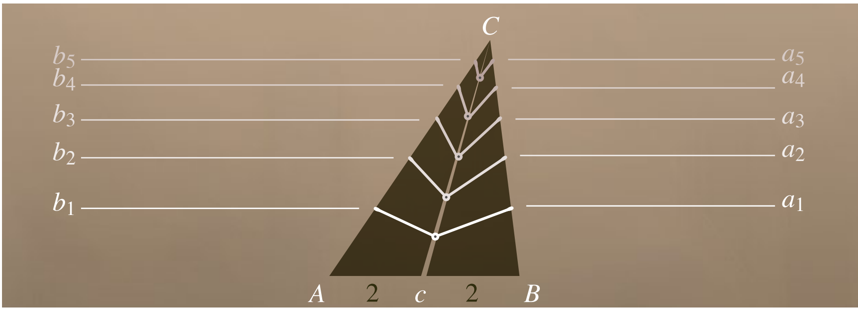
Let's experiment. Here is a triangle $\triangle ABC$ with sides of length four, five, and six.

$$|AB| = 4 \quad |BC| = 5 \quad |AC| = 6.$$



As an easy initial case, let's say that one of the three segments, say Cc, is a median (in other words, that c is the midpoint of AB). Now work backwards. Say that the triple of segments in question are concurrent. That concurrence could happen anywhere along Cc, so I have chosen five points P_i to serve as our sample points of concurrence. Once those points of concurrence have been chosen, that determines the other two segments–one passes through A and P_i, the other through B and P_i. I am interested in where those segments cut the sides of $\triangle ABC$. Label:

b_i: the intersection of BP_i and AC
a_i: the intersection of AP_i and BC



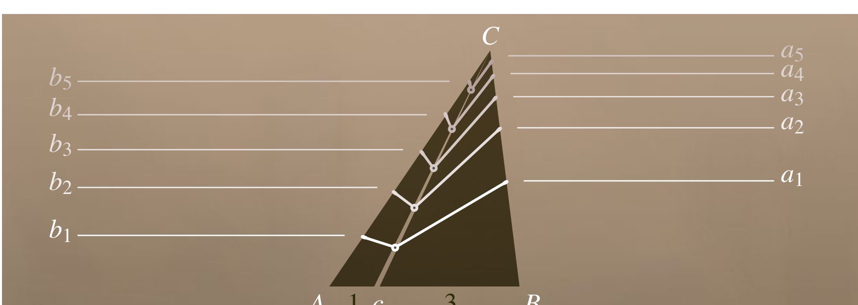
Here are the measurements (two decimal place accuracy):

i :	1	2	3	4	5
$\lvert Ab_i\rvert$	1.71	3.00	4.00	4.80	5.45
$\lvert Cb_i\rvert$	4.29	3.00	2.00	1.20	0.55
$\lvert Ba_i\rvert$	1.43	2.50	3.33	4.00	4.55
$\lvert Ca_i\rvert$	3.57	2.50	1.67	1.00	0.45

Out of all of that it may be difficult to see a useful pattern, but compare the ratios of the sides $|Ab_i|/|Cb_i|$ and $|Ba_i|/|Ca_i|$ (after all, similarity is all about ratios).

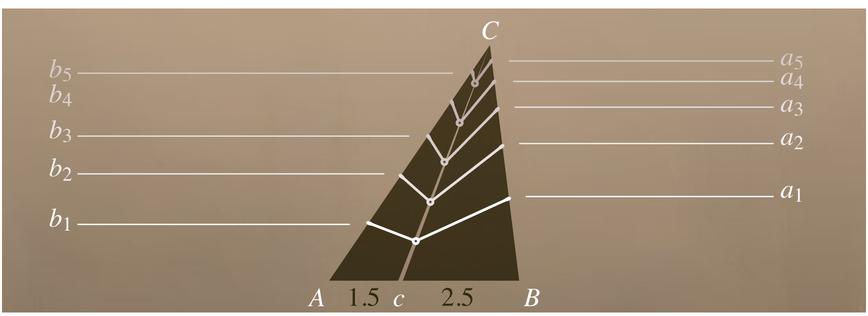
i :	1	2	3	4	5
$\lvert Ab_i\rvert/\lvert Cb_i\rvert$	0.40	1.00	2.00	4.00	10.00
$\lvert Ba_i\rvert/\lvert Ca_i\rvert$	0.40	1.00	2.00	4.00	10.00

They are the same! Let's not jump the gun though– what if Cc isn't a median? For instance, let's reposition c so that it is a distance of one from A and three from B.



$i:$	1	2	3	4	5
$\lvert Ab_i\rvert$	1.26	2.40	3.43	4.36	5.22
$\lvert Cb_i\rvert$	4.74	3.60	2.57	1.64	0.78
$\lvert Ba_i\rvert$	2.22	3.33	4.00	4.45	4.76
$\lvert Ca_i\rvert$	2.78	1.67	1.00	0.55	0.24
$\lvert Ab_i\rvert/\lvert Cb_i\rvert$	0.27	0.67	1.33	2.67	6.67
$\lvert Ba_i\rvert/\lvert Ca_i\rvert$	0.80	2.00	4.00	8.02	20.12

The ratios are not the same. Look carefully, though– the ratios $|Ba_i|/|Ca_i|$ are always three times the corresponding ratios $|Ab_i|/|Cb_i|$ (other than a bit of round-off error). Interestingly, that is the same as the ratio $|Bc|/|Ac|$. Let's do one more example, with $|Ac| = 1.5$ and $|Bc| = 2.5$.



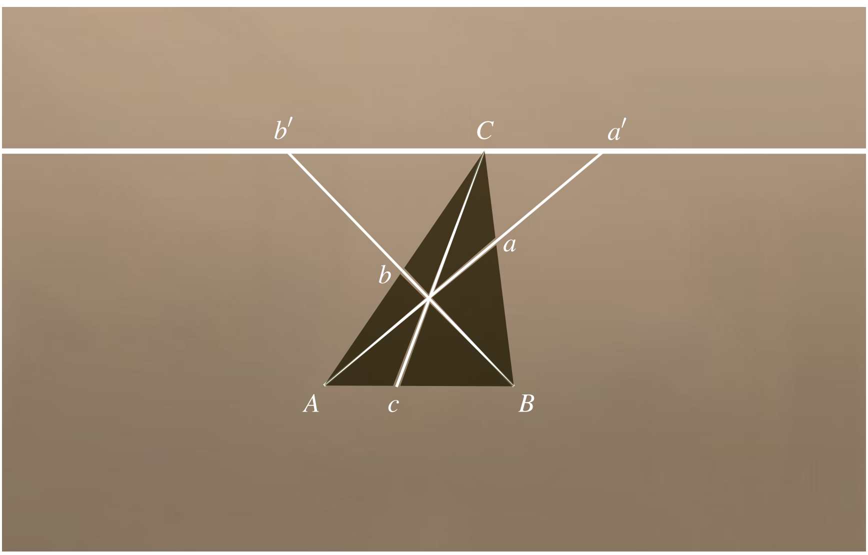
$i:$	1	2	3	4	5
$\lvert Ab_i\rvert$	1.45	2.67	3.69	4.57	5.33
$\lvert Cb_i\rvert$	4.55	3.33	2.31	1.43	0.67
$\lvert Ba_i\rvert$	1.74	2.86	3.64	4.21	4.65
$\lvert Ca_i\rvert$	3.26	2.14	1.36	0.79	0.35
$\lvert Ab_i\rvert/\lvert Cb_i\rvert$	0.32	0.80	1.60	3.20	8.00
$\lvert Ba_i\rvert/\lvert Ca_i\rvert$	0.53	1.33	2.66	5.34	13.33

Once again, the ratios $|Ab_i|/|Cb_i|$ all hover about 1.67, right at the ratio $|Bc|/|Ac|$. What we have stumbled across is called Ceva's Theorem, but it is typically given a bit more symmetrical presentation.

CEVA'S THEOREM

Three segments Aa, Bb, and Cc, that connect the vertices of $\triangle ABC$ to their respective opposite sides, are concurrent if and only if

$$\frac{|Ab|}{|bC|} \cdot \frac{|Ca|}{|aB|} \cdot \frac{|Bc|}{|cA|} = 1.$$



Proof. $\Longrightarrow$ Similar triangles anchor this proof. To get to those similar triangles, though, we need to extend the illustration a bit. Assume that Aa, Bb, and Cc concur at a point P. Draw out the line which passes through C and is parallel to AB; then extend Aa and Bb so that they intersect this line. Mark those intersection points as a' and b' respectively. We need to look at four pairs of similar triangles.

They are:

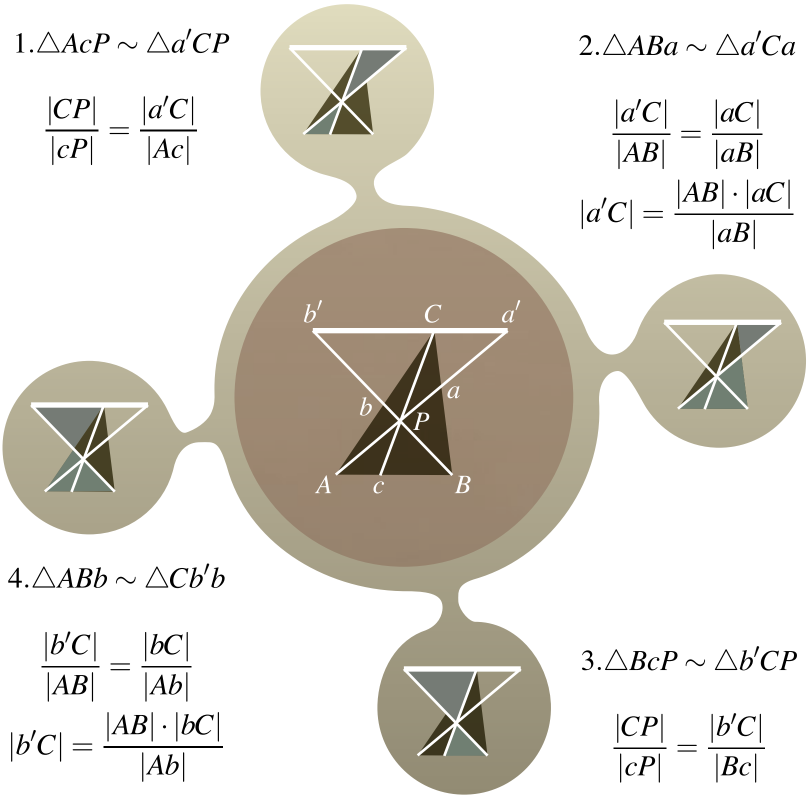
1. $\triangle AcP \sim \triangle a'CP$

$$\frac{|CP|}{|cP|} = \frac{|a'C|}{|Ac|}$$

2. $\triangle ABa \sim \triangle a'Ca$

$$\frac{|a'C|}{|AB|} = \frac{|aC|}{|aB|}$$

$$|a'C| = \frac{|AB| \cdot |aC|}{|aB|}$$



4. $\triangle ABb \sim \triangle Cb'b$

$$\frac{|b'C|}{|AB|} = \frac{|bC|}{|Ab|}$$

$$|b'C| = \frac{|AB| \cdot |bC|}{|Ab|}$$

3. $\triangle BcP \sim \triangle b'CP$

$$\frac{|CP|}{|cP|} = \frac{|b'C|}{|Bc|}$$

Plug the second equation into the first

$$\frac{|CP|}{|cP|} = \frac{|AB| \cdot |aC|}{|aB| \cdot |Ac|}$$

and the fourth into the third

$$\frac{|CP|}{|cP|} = \frac{|AB| \cdot |bC|}{|Ab| \cdot |BC|}$$

Set these two equations equal and simplify

$$\frac{|AB| \cdot |aC|}{|aB| \cdot |Ac|} = \frac{|AB| \cdot |bC|}{|Ab| \cdot |BC|} \implies \frac{|Ab|}{|bC|} \cdot \frac{|Ca|}{|aB|} \cdot \frac{|Bc|}{|cA|} = 1.$$

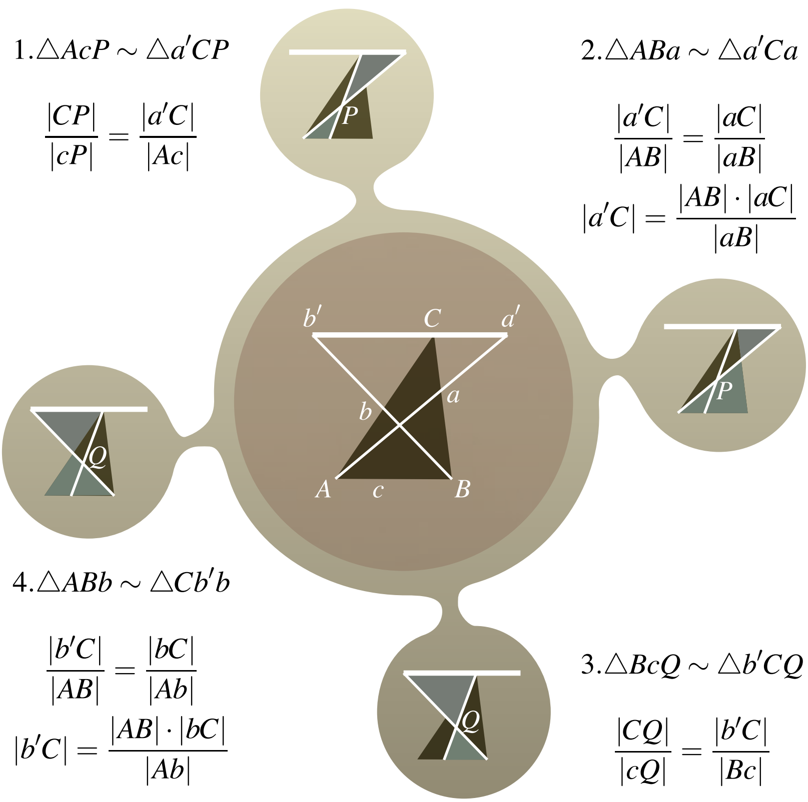
$\Longleftarrow$ A similar tactic works for the other direction. For this part, we are going to assume the equation

$$\frac{|Ab|}{|bC|}\cdot\frac{|Ca|}{|aB|}\cdot\frac{|Bc|}{|cA|}=1,$$

and show that Aa, Bb, and Cc are concurrent. Label

P: the intersection of Aa and Cc
Q: the intersection of Bb and Cc.

In order for all three segments to concur, P and Q will actually have to be the same point. We can show that they are by computing the ratios $|AP|/|aP|$ and $|AQ|/|aQ|$ and seeing that they are equal. That will mean that P and Q have to be the same distance down the segment Aa from A, and thus guarantee that they are the same. Again with the similar triangles:



1. $\triangle AcP \sim \triangle a'CP$

$$\frac{|CP|}{|cP|}=\frac{|a'C|}{|Ac|}$$

2. $\triangle ABa \sim \triangle a'Ca$

$$\frac{|a'C|}{|AB|}=\frac{|aC|}{|aB|}$$

$$|a'C|=\frac{|AB|\cdot|aC|}{|aB|}$$

3. $\triangle BcQ \sim \triangle b'CQ$

$$\frac{|CQ|}{|cQ|}=\frac{|b'C|}{|Bc|}$$

4. $\triangle ABb \sim \triangle Cb'b$

$$\frac{|b'C|}{|AB|}=\frac{|bC|}{|Ab|}$$

$$|b'C|=\frac{|AB|\cdot|bC|}{|Ab|}$$

Plug the second equation into the first

$$\frac{|CP|}{|cP|} = \frac{|aC| \cdot |AB|}{|aB| \cdot |Ac|}$$

and the fourth equation into the third

$$\frac{|CQ|}{|cQ|} = \frac{|AB| \cdot |bC|}{|Ab| \cdot |Bc|}$$

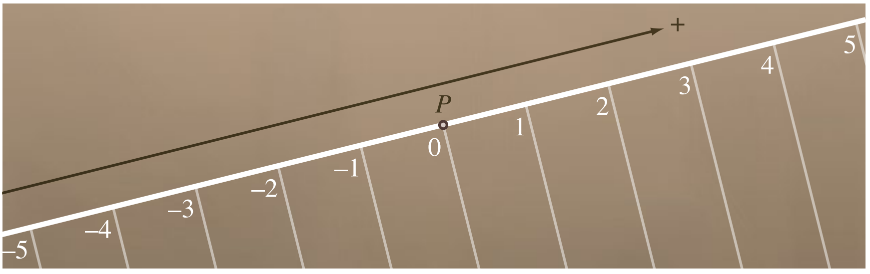
Now divide and simplify

$$\frac{|CP|}{|cP|} \Big/ \frac{|CQ|}{|cQ|} = \frac{|aC| \cdot |AB| \cdot |Ab| \cdot |Bc|}{|aB| \cdot |Ac| \cdot |AB| \cdot |bC|} = \frac{|Ab|}{|bC|} \cdot \frac{|Ca|}{|aB|} \cdot \frac{|Bc|}{|cA|} = 1.$$

Therefore $|AP|/|aP| = |AQ|/|aQ|$, so $P = Q$. □

Ceva's Theorem is great for concurrences inside the triangle, but we have seen that concurrences can happen outside the triangle as well (such as the orthocenter of an obtuse triangle). Will this calculation still tell us about those concurrences? Well, not quite. If the three lines concur, then the calculation will still be one, but now the calculation can mislead– it is possible to calculate one when the lines do not concur. If you look back at the proof, you can see the problem. If P and Q are on the opposite side of a, then the ratios $|AP|/|aP|$ and $|AQ|/|aQ|$ could be the same even though $P \neq Q$. There is a way to repair this, though. The key is "signed distance". We assign to each of the three lines containing a side of the triangle a direction (saying this way is positive, this way is negative). For two points A and B on one of those lines, the signed distance is defined as

$$[AB] = \begin{cases} |AB| & \text{if the ray } AB\rightarrow \text{ points in the positive direction} \\ -|AB| & \text{if the ray } AB\rightarrow \text{ points in the negative direction.} \end{cases}$$



Signed distance from P. The sign is determined by a choice of direction.

This simple modification is all that is needed to extend Ceva's Theorem

CEVA'S THEOREM (EXTENDED VERSION)
Three lines Aa, Bb, and Cc, that connect the vertices of $\triangle ABC$ to the lines containing their respective opposite sides, are concurrent if and only if

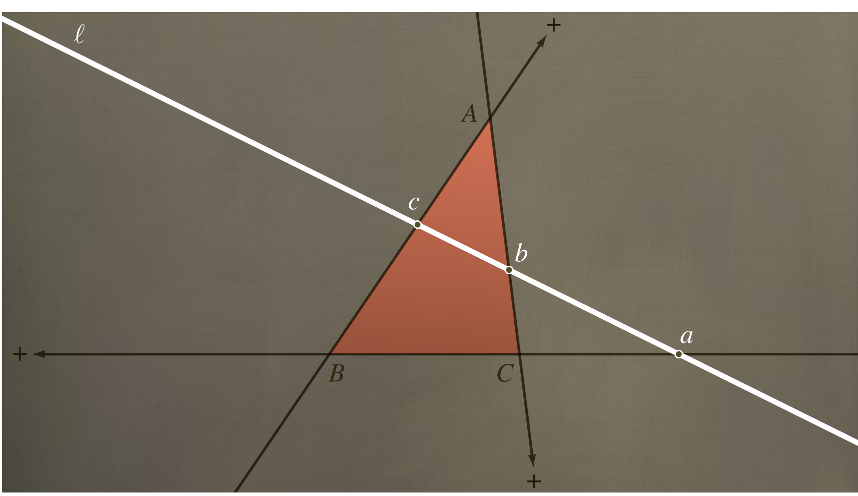
$$\frac{[Ab]}{[bC]} \cdot \frac{[Ca]}{[aB]} \cdot \frac{[Bc]}{[cA]} = 1.$$

Menelaus's Theorem

Ceva's Theorem is one of a pair– the other half is its projective dual, Menelaus's Theorem. We are not going to look at projective geometry in this book, but one of its key underlying concepts is that at the level of incidence, there is a duality between points and lines. For some very fundamental results, this duality allows the roles of the two to be interchanged.

MENELAUS'S THEOREM
For a triangle $\triangle ABC$, and points a on $\leftarrow BC \rightarrow$, b on $\leftarrow AC \rightarrow$, and c on $\leftarrow AB \rightarrow$, a, b, and c are colinear if and only if

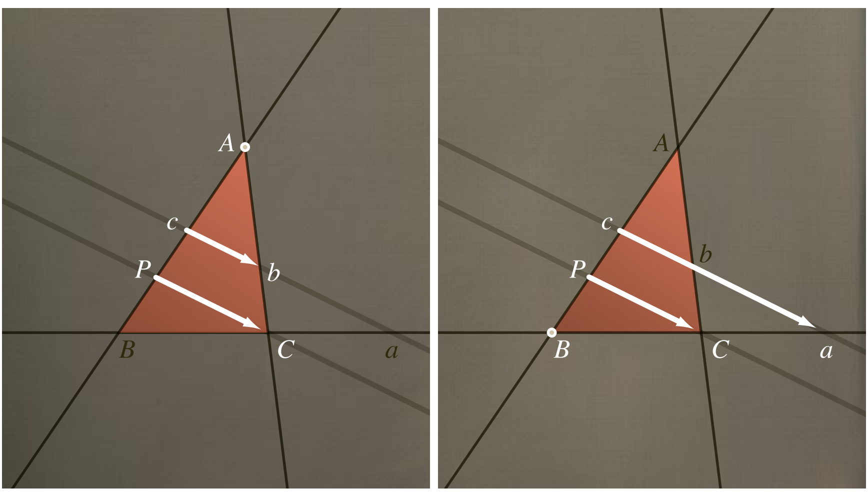
$$\frac{[Ab]}{[bC]} \cdot \frac{[Ca]}{[aB]} \cdot \frac{[Bc]}{[cA]} = -1.$$



Proof. $\Longrightarrow$ Suppose that $a, b,$ and c all lie along a line ℓ. The requirement that $a, b,$ and c all be distinct prohibits any of the three intersections from occurring at a vertex. According to Pasch's Lemma, then, ℓ will intersect two sides of the triangle, or it will miss all three sides entirely. Either way, it has to miss one of the sides. Let's say that missed side is BC. There are two ways this can happen:

1. ℓ intersects line BC on the opposite side of B from C
2. ℓ intersects line BC on the opposite side of C from B

The two scenarios will play out very similarly, so let's just look at the second one. Draw the line through C parallel to ℓ. Label its intersection with AB as P. That sets up some useful parallel projections.



From AB to AC:

$$A \mapsto A \quad c \mapsto b \quad P \mapsto C.$$

Comparing ratios,

$$\frac{|cP|}{|bC|} = \frac{|Ac|}{|Ab|}$$

and so

$$|cP| = \frac{|Ac|}{|Ab|} \cdot |bC|.$$

From AB to BC:

$$B \mapsto B \quad c \mapsto a \quad P \mapsto C.$$

Comparing ratios,

$$\frac{|cP|}{|aC|} = \frac{|Bc|}{|Ba|}$$

and so

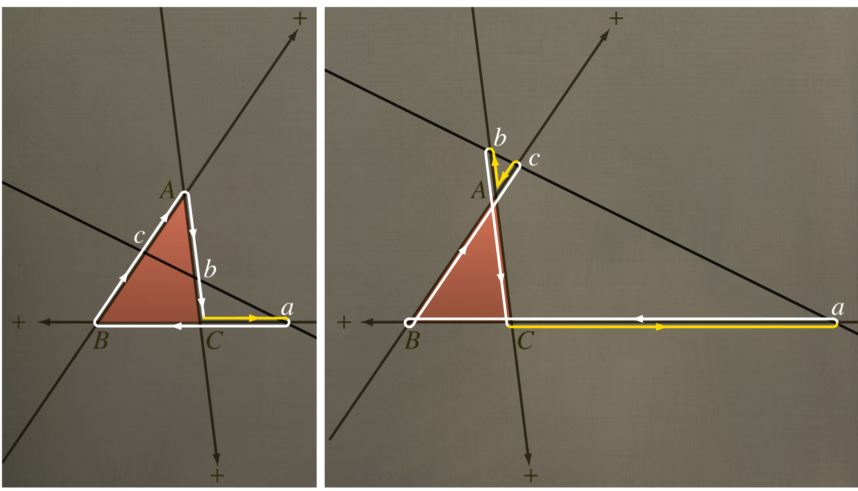
$$|cP| = \frac{|Bc|}{|Ba|} \cdot |aC|.$$

Just divide the second $|cP|$ by the first $|cP|$ to get

$$1 = \frac{|cP|}{|cP|} = \frac{|Ab|\cdot|aC|\cdot|Bc|}{|Ac|\cdot|bC|\cdot|Ba|} = \frac{|Ab|}{|bC|}\cdot\frac{|Ca|}{|aB|}\cdot\frac{|Bc|}{|cA|}.$$

That's close, but we are after an equation that calls for *signed* distance. So orient the three lines of the triangle so that $AC\to$, $CB\to$, and $BA\to$ all point in the positive direction (any other orientation will flip pairs of signs that will cancel each other out). With this orientation, if ℓ intersects two sides of the triangle, then all the signed distances involved are positive except $[Ca] = -|Ca|$. If ℓ misses all three sides of the triangle, then three of the signed distances are positive, but three are not:

$$[Ab] = -|Ab| \quad [Ca] = -|Ca| \quad [cA] = -|cA|.$$

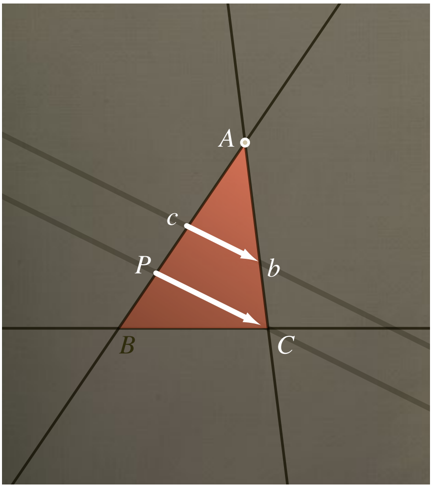


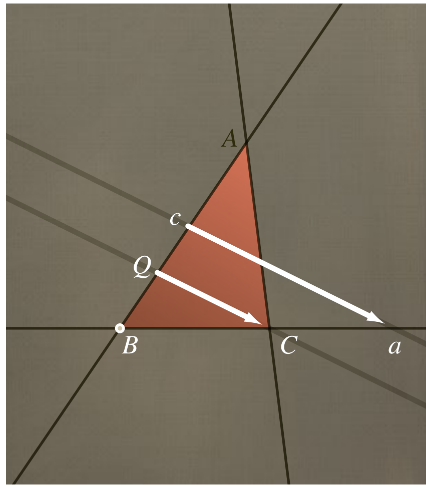
Either way, an odd number of signs are changed, so

$$\frac{[Ab]}{[bC]}\frac{[Ca]}{[aB]}\frac{[Bc]}{[cA]} = -1.$$

$\Longleftarrow$ Let's turn the argument around to prove the converse. Suppose that

$$\frac{[Ab]}{[bC]}\cdot\frac{[Ca]}{[aB]}\cdot\frac{[Bc]}{[cA]} = -1.$$





Draw the line from C that is parallel to bc and label its intersection with AB as P. There is a parallel projection from AB to AC so that

$$A \mapsto A \quad c \mapsto b \quad P \mapsto C$$

and therefore

$$\frac{|cP|}{|Ac|} = \frac{|Cb|}{|bA|}.$$

Draw the line from C that is parallel to ac, and label its intersection with AB as Q. There is a parallel projection from AB to BC so that

$$B \mapsto B \quad c \mapsto a \quad Q \mapsto C$$

and therefore

$$\frac{|cQ|}{|cB|} = \frac{|Ca|}{|Ba|}$$

Now solve those equations for $|cP|$ and $|cQ|$, and divide to get

$$\frac{[cQ]}{[cP]} = \frac{[bA]\cdot[Ca]\cdot[cB]}{[Cb]\cdot[Ac]\cdot[Ba]} = -\frac{[Ab]}{[bC]}\cdot\frac{[Ca]}{[aB]}\cdot\frac{[Bc]}{[cA]} = -(-1) = 1.$$

Both P and Q are the same distance from c along cC. That means they must be the same. $\square$

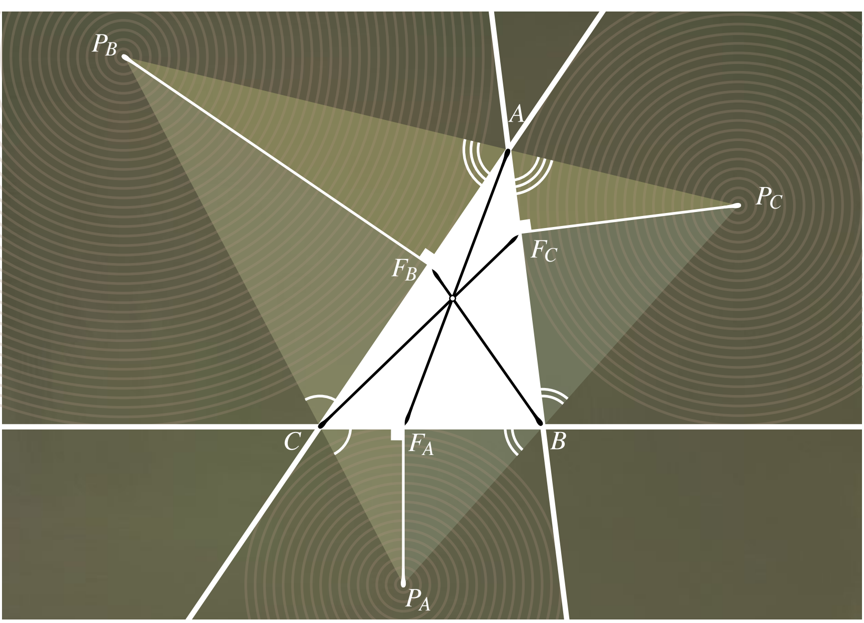
The Nagel point

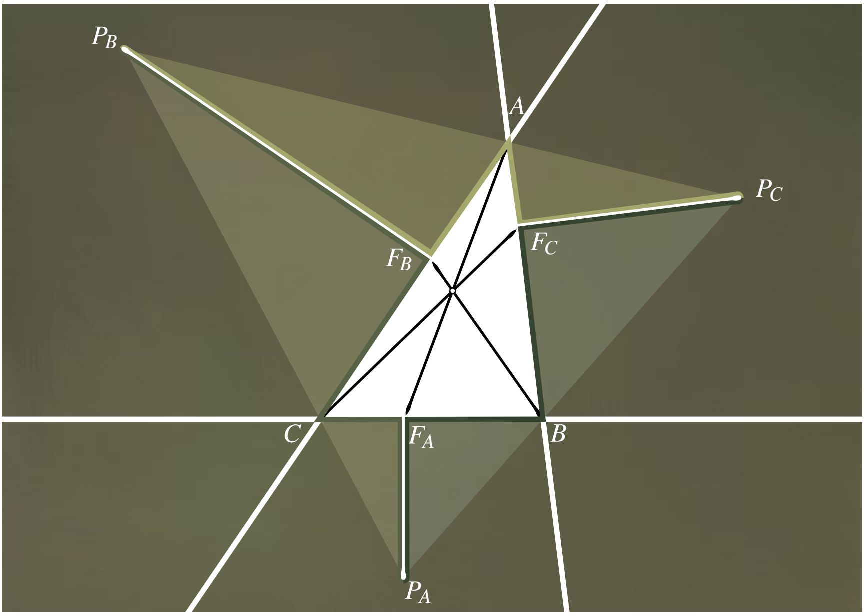
Back to excircles for one more concurrence, and this time we will use Ceva's Theorem to prove it.

THE NAGEL POINT
If $\mathcal{C}_A$, $\mathcal{C}_B$, and $\mathcal{C}_C$ are the three excircles of a triangle $\triangle ABC$ so that $\mathcal{C}_A$ is in the interior of $\angle A$, $\mathcal{C}_B$ is in the interior of $\angle B$, and $\mathcal{C}_C$ is in the interior of $\angle C$; and if F_A is the intersection of $\mathcal{C}_A$ with BC, F_B is the intersection of $\mathcal{C}_B$ with AC, and F_C is the intersection of $\mathcal{C}_C$ with AB; then the three segments AF_A, BF_B, and CF_C are concurrent. This point of concurrence is called the Nagel point.

Proof. This is actually pretty easy thanks to Ceva's Theorem. The key is similar triangles. Label P_A, the center of excircle $\mathcal{C}_A$, P_B, the center of excircle $\mathcal{C}_B$, and P_C, the center of excircles, $\mathcal{C}_C$. By A·A triangle similarity,

$$\triangle P_AF_AC \sim \triangle P_BF_BC$$
$$\triangle P_BF_BA \sim \triangle P_CF_CA$$
$$\triangle P_CF_CB \sim \triangle P_AF_AB.$$





Ceva's Theorem promises concurrence if we can show that

$$\frac{|AF_C|}{|F_CB|} \cdot \frac{|BF_A|}{|F_AC|} \cdot \frac{|CF_B|}{|F_BA|} = 1.$$

Those triangle similarities give some useful ratios to that end:

$$\frac{|AF_C|}{|AF_B|} = \frac{|P_CF_C|}{|P_BF_B|} \qquad \frac{|BF_A|}{|BF_C|} = \frac{|P_AF_A|}{|P_CF_C|} \qquad \frac{|CF_B|}{|CF_A|} = \frac{|P_BF_B|}{|P_AF_A|}.$$

So

$$\begin{aligned} \frac{|AF_C|}{|F_CB|} \frac{|BF_A|}{|F_AC|} \frac{|CF_B|}{|F_BA|} &= \frac{|AF_C|}{|AF_B|} \frac{|BF_A|}{|BF_C|} \frac{|CF_B|}{|CF_A|} \\ &= \frac{|P_CF_C|}{|P_BF_B|} \frac{|P_AF_A|}{|P_CF_C|} \frac{|P_BF_B|}{|P_AF_A|} \\ &= 1. \end{aligned}$$

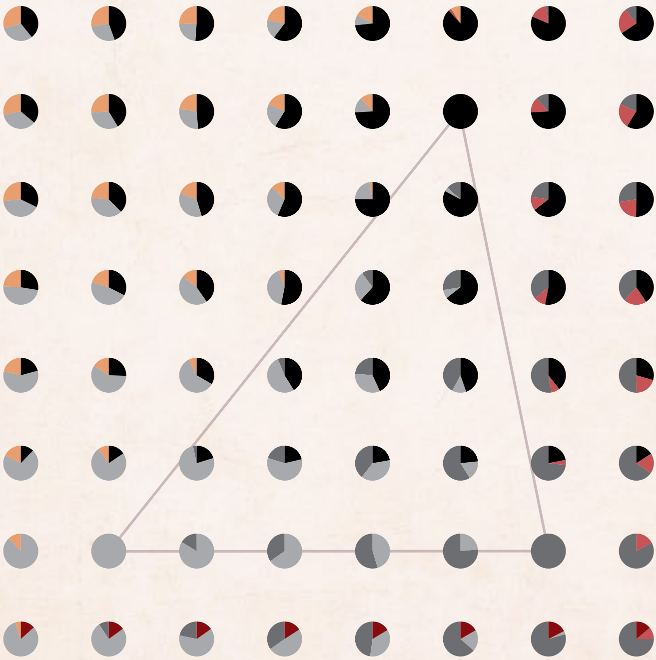
By Ceva's Theorem, the three segments are concurrent. □

Exercises

1. Use Ceva's Theorem to prove that the medians of a triangle are concurrent.

2. Use Ceva's Theorem to prove that the orthocenters of a triangle are concurrent.

3. Give a compass and straight-edge construction of the three excircles and the nine-point circle of a given triangle. If your construction is accurate enough, you should notice that the excircles are all tangent to the nine-point circle (a result commonly called Feuerbach's Theorem).



22 TRILINEAR COORDINATES

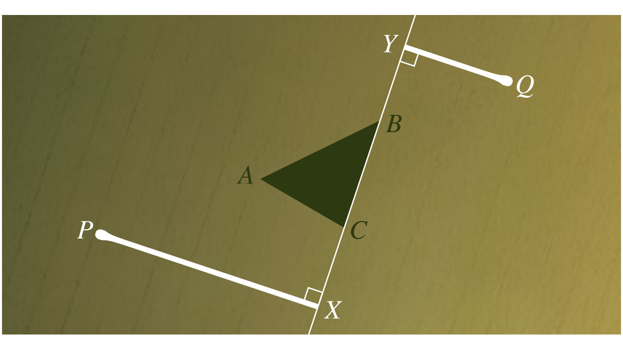
This is my last lesson under the heading of "Euclidean geometry". If you look back to the start, we have built a fairly impressive structure from modest beginnings. Throughout it all, I have aspired to a synthetic approach to the subject, which is to say that I have avoided attaching a coordinate system to the plane, with all the powerful analytic techniques that come by doing so. I feel that it is in the classical spirit of the subject to try to maintain this synthetic stance for as long as possible. But as we now move into the more modern development of the subject, it is time to shift positions. As a result, much of the rest of this work will take on a decidedly different flavor. With this lesson, I hope to capture the inflection point of that shift in stance, from the synthetic to the analytic.

Trilinear coordinates

In this lesson, we will look at trilinear coordinates, a coordinate system that is closely tied to the concurrence results of the last few lessons. Essentially, trilinear coordinates are defined by measuring signed distances from the sides of a given triangle.

DEF: THE SIGNED DISTANCE TO A SIDE OF A TRIANGLE
Given a side s of a triangle $\triangle ABC$ and a point P, let $|P,s|$ denote the (minimum) distance from P to the line containing s. Then define the signed distance from P to s as

$$[P,s] = \begin{cases} |P,s| & \text{if } P \text{ is on the same side of } s \text{ as the triangle} \\ -|P,s| & \text{if } P \text{ is on the opposite side of } s \text{ from the triangle} \end{cases}$$



$[P,BC] = |PX|$
$[Q,BC] = -|QY|$

From these signed distances, every triangle creates a kind of coordinate system in which a point P in the plane is assigned three coordinates

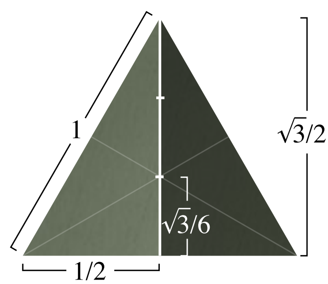
$$\alpha = [P,BC] \quad \beta = [P,AC] \quad \gamma = [P,AB].$$

This information is consolidated into the notation $P = [\alpha : \beta : \gamma]$. There is an important thing to notice about this system of coordinates: while every point corresponds to a triple of real numbers, not every triple of real numbers corresponds to a point. For instance, when $\triangle ABC$ is equilateral with sides of length one, there is no point with coordinates $[2:2:2]$. Fortunately, there is a way around this limitation, via an equivalence relation.

AN EQUIVALENCE RELATION ON COORDINATES
Two sets of trilinear coordinates $[a:b:c]$ and $[d:b':c']$ are equivalent, written $[a:b:c] \sim [d':b':c']$, if there is a real number $k \neq 0$ so that

$$a' = ka \quad b' = kb \quad c' = kc.$$

Consider again that equilateral triangle $\triangle ABC$ with sides of length one. Okay, there is no point which is a distance of two from each side. But $[2:2:2]$ is equivalent to $[\sqrt{3}/6 : \sqrt{3}/6 : \sqrt{3}/6]$, and there is a point which is a distance of $\sqrt{3}/6$ from each side– the center of the triangle. That brings us to the definition of trilinear coordinates.



DEF: TRILINEAR COORDINATES
The trilinear coordinates of a point P with respect to a triangle $\triangle ABC$ is the equivalence class of triples $[k\alpha : k\beta : k\gamma]$ (with $k \neq 0$) where

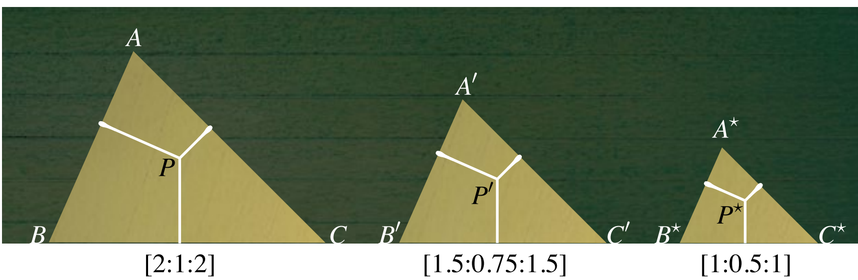
$$\alpha = [P,BC] \quad \beta = [P,AC] \quad \gamma = [P,AB].$$

The coordinates corresponding to the actual signed distances, when $k = 1$, are called the exact trilinear coordinates of P.

Because each coordinate is actually an equivalence class, there is an immediately useful relationship between trilinear coordinates in similar triangles. Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are similar, with a scaling constant k so that

$$|A'B'| = k|AB| \quad |B'C'| = k|BC| \quad |C'A'| = k|CA|.$$

Suppose that P and P' are points that are positioned similarly with respect to those triangles (so that $|A'P'| = k|AP|$, $|B'P'| = k|BP|$, and $|C'P'| = k|CP|$). Then the coordinates of P as determined by $\triangle ABC$ will be equivalent to the coordinates of P' as determined by $\triangle A'B'C'$.



Exact trilinear coordinates of similarly positioned points in similar triangles.

With that in mind, let's get back to the question of whether every *equivalence class* of triples of real numbers corresponds to a point. Straight out of the gate, the answer is no– the coordinates $[0:0:0]$ do not correspond to any point. As it turns out, that is the exception.

THM: THE RANGE OF THE TRILINEARS

Given a triangle $\triangle ABC$ and real numbers x, y, and z, not all zero, there is a point whose trilinear coordinates with respect to $\triangle ABC$ are $[x:y:z]$.

Proof. There are essentially two cases: one where all three of x, y, and z have the same sign, and one where they do not. I will look at the first case in detail. The second differs at just one crucial step, so I will leave the details of that case to you. In both cases, my approach is a constructive one, but it does take a rather indirect path. Instead of trying to find a point inside $\triangle ABC$ with the correct coordinates, I will start with a point P, and then build a new triangle $\triangle abc$ around it.

That new triangle will

1. be similar to the original $\triangle ABC$, and
2. be positioned so that the trilinear coordinates of P with respect to $\triangle abc$ are $[x : y : z]$.

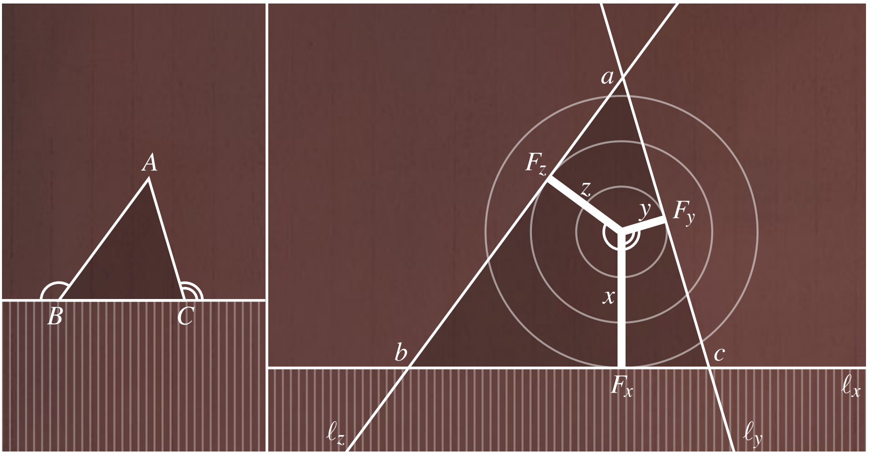
Then the similarly positioned point in $\triangle ABC$ will have to have those same coordinates relative to $\triangle ABC$.

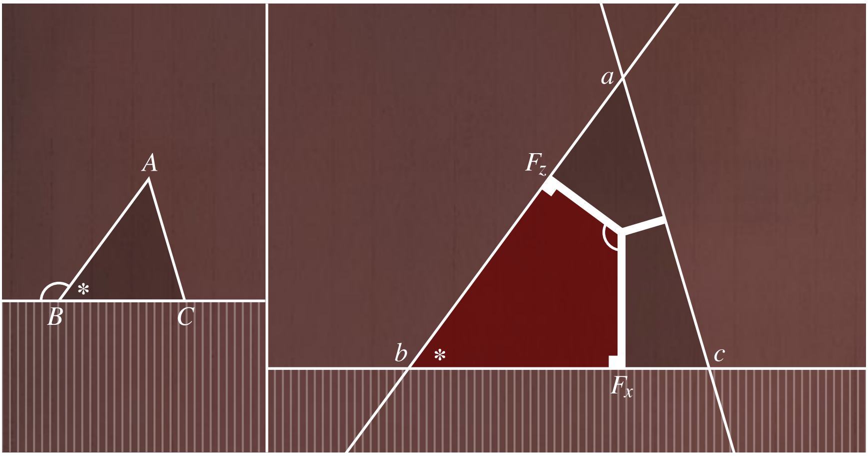
Case 1. $[+ : + : +] \sim [- : - : -]$
Consider the situation where all three numbers x, y, and z are greater than or equal to zero (of course, they cannot all be zero, since a point cannot be on all three sides of a triangle). This also handles the case where all three coordinates are negative, since $[x : y : z] \sim [-x : -y : -z]$. Mark a point F_x which is a distance x away from P. On opposite sides of the ray $PF_x \rightarrow$, draw out two more rays to form angles measuring $\pi - (\angle B)$ and $\pi - (\angle C)$. On the first ray, mark the point F_z which is a distance z from P. On the second, mark the point F_y which is a distance y from P. Let

ℓ_x be the line through F_x that is perpendicular to PF_x,
ℓ_y be the line through F_y that is perpendicular to PF_y,
ℓ_z be the line through F_z that is perpendicular to PF_z.

Label their points of intersection as

$$a = \ell_y \cap \ell_z \quad b = \ell_x \cap \ell_z \quad c = \ell_x \cap \ell_y.$$





Clearly, the trilinear coordinates of P relative to $\triangle abc$ are $[x : y : z]$. To see that $\triangle abc$ and $\triangle ABC$ are similar, let's compare their interior angles. The quadrilateral PF_xbF_z has right angles at vertex F_x and F_z and an angle measuring $\pi - (\angle B)$ at vertex P. Since the angle sum of a quadrilateral is 2π, that means $(\angle b) = (\angle B)$, so they are congruent. By a similar argument, $\angle c$ and $\angle C$ must be congruent. By A·A similarity, then, $\triangle ABC$ and $\triangle abc$ are similar.

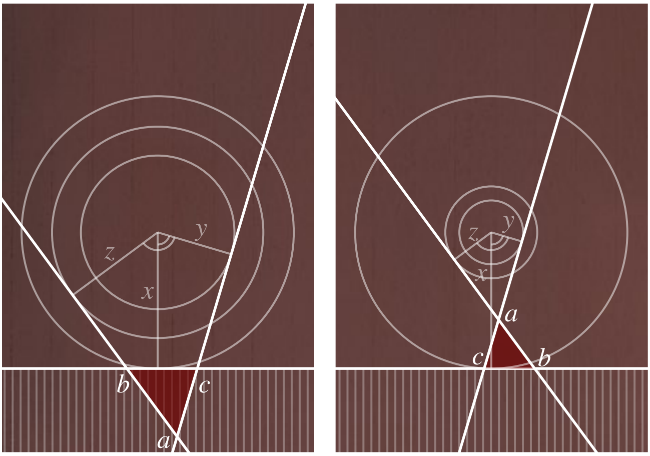
Case 2. $[+ : - : -] \sim [- : + : +]$
Other than some letter shuffling, this also handles scenarios of the form $[- : + : -]$, $[+ : - : +]$, $[- : - : +]$, and $[+ : + : -]$. Use the same construction as in the previous case, but with one important change: in the previous construction, we needed

$$(\angle F_zPF_x) = \pi - (\angle B) \quad \& \quad (\angle F_yPF_x) = \pi - (\angle C).$$

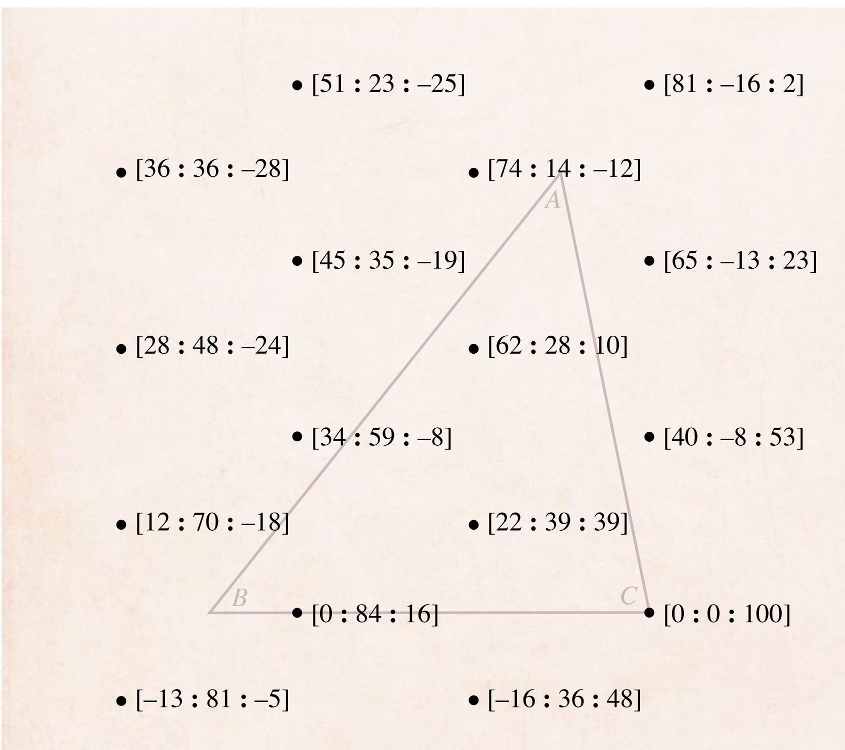
This time we are going to want

$$(\angle F_zPF_x) = (\angle B) \quad \& \quad (\angle F_yPF_x) = (\angle C).$$

The construction still forms a triangle $\triangle abc$ that is similar to $\triangle ABC$, but now P lies outside of it. Depending upon the location of a relative to the line ℓ_x, the signed distances from P to BC, AC, and AB, respectively are either x, y, and z, or $-x$, $-y$ and $-z$. Either way, since $[x : y : z]$ is equivalent to $[-x : -y : -z]$, P has the correct coordinates. $\square$



Case 2.
(l) exact trilinears with form [–:+:+]
(r) exact trilinears with form [+:–:–]



Trilinear coordinates of a few points, normalized so that the sum of the magnitudes of the coordinates is 100, and rounded to the nearest integer.

Trilinears of the classical centers

The classical triangle centers that we have studied in the last few lessons tend to have elegant trilinear coordinates. The rest of this lesson is dedicated to finding a few of them. The easiest of these, of course, is the incenter. Since it is equidistant from each of the three sides of the triangle, its trilinear coordinates are $[1:1:1]$. The others will require a little bit more work. These formulas are valid for all triangles, but if $\triangle ABC$ is obtuse, then one of its angles is obtuse, and thus far we have only really discussed the trigonometry of acute angles. For that reason, in these proofs I will restrict my attention to acute triangles. Of course, you have surely seen the unit circle extension of the trigonometric functions to all angle measures, so I encourage you to complete the proof by considering triangles that are not acute.

TRILINEARS OF THE CIRCUMCENTER
The trilinear coordinates of the circumcenter of $\triangle ABC$ are

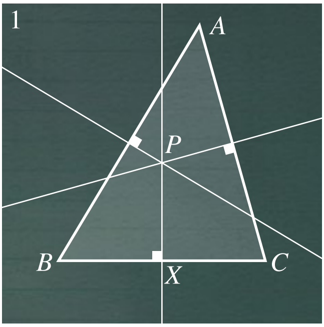
$$[\cos A : \cos B : \cos C].$$

Proof. First the labels. Label the circumcenter P. Recall that the circumcenter is the intersection of the perpendicular bisectors of the three sides of the triangle. Let's take just one of those: the perpendicular bisector to BC. It intersects BC at its midpoint– call that point X. Now we can calculate the first exact trilinear coordinate in just a few steps, which I will justify below.

$$[P,BC] \underset{①}{=} |PX| \underset{②}{=} |PB|\cos(\angle BPX) \underset{③}{=} |PB|\cos(\angle BAC).$$

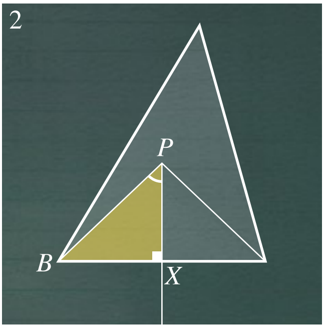
1. The minimum distance from P to BC is along the perpendicular– so $|P,BC| = |P,X|$. We have assumed that $\triangle ABC$ is acute. That places P inside the triangle, on the same side of BC as A, which means that the signed distance $[P,BC]$ is positive. Therefore

$$[P,BC] = |P,BC| = |PX|.$$



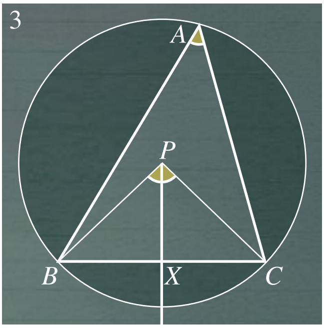
2. Look at $\angle BPX$ in the triangle $\triangle BPX$:

$$\cos(\angle BPX) = \frac{|PX|}{|PB|}$$
$$\implies |PX| = |PB|\cos(\angle BPX).$$



3. Segment PX splits $\triangle BPC$ into two pieces, $\triangle BPX$ and $\triangle CPX$, which are congruent by S·A·S. Thus PX evenly divides the *angle* $\angle BPC$ into two congruent pieces, and so

$$(\angle BPX) = \tfrac{1}{2}(\angle BPC).$$



Recall that the circumcenter is the center of the circle which passes through all three vertices A, B, and C. With respect to that circle, $\angle BAC$ is an inscribed angle, and $\angle BPC$ is the corresponding central angle. According to the Inscribed Angle Theorem,

$$(\angle BAC) = \tfrac{1}{2}(\angle BPC).$$

That means that $(\angle BPX) = (\angle BAC)$.

With that same argument we can find the signed distances to the other two sides as well.

$$[P, AC] = |PC|\cos(\angle ABC) \quad \& \quad [P, AB] = |PA|\cos(\angle BCA)$$

Gather that information together to get the exact trilinear coordinates of the circumcenter

$$P = [|PB|\cos(\angle A) : |PC|\cos(\angle B) : |PA|\cos(\angle C)].$$

Finally, observe that PA, PB, and PC are all the same length– they are radii of the circumcircle. Therefore, we can factor out that constant to get an equivalent representation

$$P = [\cos(\angle A) : \cos(\angle B) : \cos(\angle C)].$$

$\square$

TRILINEARS OF THE ORTHOCENTER

The trilinear coordinates of the orthocenter of $\triangle ABC$ are

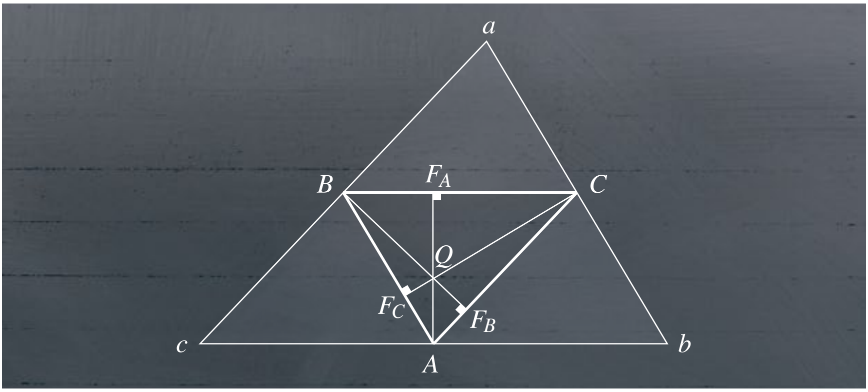
$$[\cos B \cos C : \cos A \cos C : \cos A \cos B].$$

Proof. Label the orthocenter Q. Recall that it is the intersection of the three altitudes of the triangle. Label the feet of those altitudes

F_A: the foot of the altitude through A,
F_B: the foot of the altitude through B, and
F_C: the foot of the altitude through C.

Now think back to the way we proved that the altitudes concur in lesson 19– it was by showing that they are the perpendicular bisectors of a larger triangle $\triangle abc$, where

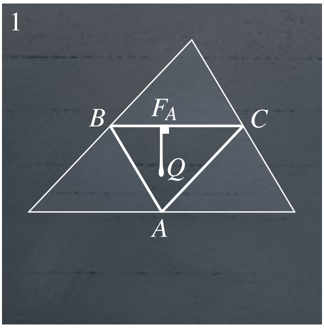
bc passed through A and was parallel to BC,
ac passed through B and was parallel to AC, and
ab passed through C and was parallel to AB.



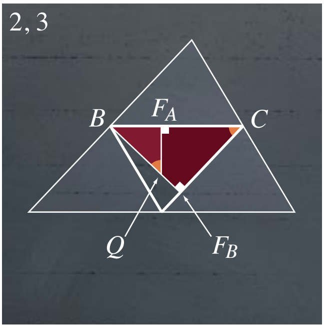
We are going to need that triangle again. Here is the essential calculation, with commentary explaining the steps below.

$$\begin{aligned}[Q,BC] &\overset{①}{=} |QF_A| \overset{②}{=} |QB|\cos(\angle F_AQB) \overset{③}{=} |QB|\cos(\angle C)\\ &\underset{④}{=} |Qa|\cos(\angle aQB)\cos(\angle C) \underset{⑤}{=} |Qa|\cos(\angle B)\cos(\angle C)\end{aligned}$$

1. The distance from Q to BC is measured along the perpendicular, so $|Q,BC| = |QF_A|$, but since we assumed our triangle is acute, Q will be inside $\triangle ABC$ and that means the signed distance $[Q,BC]$ is positive. So
$$[Q,BC] = |Q,BC| = |QF_A|.$$

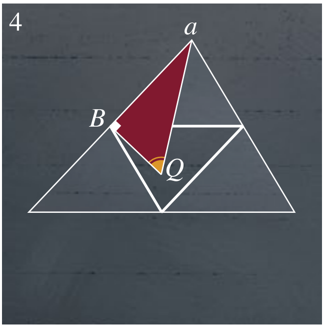


2. Look at the right triangle $\triangle F_AQB$. In it,
$$\cos(\angle F_AQB) = \frac{|QF_A|}{|QB|}$$
$$\implies |QF_A| = |QB|\cos(\angle F_AQB).$$

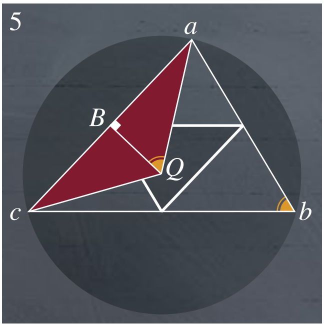


3. By A·A, $\triangle F_AQB \sim \triangle F_BCB$ (they share the angle at B and both have a right angle). Therefore
$$\angle F_AQB \simeq \angle F_BCB.$$

4. Look at the right triangle $\triangle aQB$. In it,
$$\cos(\angle aQB) = \frac{|QB|}{|Qa|}$$
$$\implies |QB| = |Qa|\cos(\angle aQB).$$



5. The orthocenter Q of $\triangle ABC$ is actually the circumcenter of the larger triangle $\triangle abc$. The angle $\angle abc$ is an inscribed angle in the circumcircle whose corresponding central angle is $\angle aQc$. By the Inscribed Angle Theorem, then,
$$(\angle abc) = \tfrac{1}{2}(\angle aQc).$$
The segment QB bisects $\angle aQc$ though, so
$$(\angle aQB) = \tfrac{1}{2}(\angle aQc).$$
That means $\angle aQB \simeq \angle abc$, which is, in turn congruent to $\angle B$ in the original triangle.



Through similar calculations,

$$[Q, AC] = |Qb| \cos(\angle A) \cos(\angle C)$$
$$[Q, AB] = |Qc| \cos(\angle A) \cos(\angle B).$$

That gives the exact trilinear coordinates for the orthocenter as

$$Q = [|Qa| \cos(\angle B) \cos(\angle C) : |Qb| \cos(\angle A) \cos(\angle C) : |Qc| \cos(\angle A) \cos(\angle B)]$$

Of course Qa, Qb and Qc are all the same length, though, since they are radii of the circumcircle of $\triangle abc$. Factoring out that constant gives an equivalent set of coordinates

$$Q = [\cos(\angle B) \cos(\angle C) : \cos(\angle A) \cos(\angle C) : \cos(\angle A) \cos(\angle B)].$$

□

TRILINEARS OF THE CENTROID

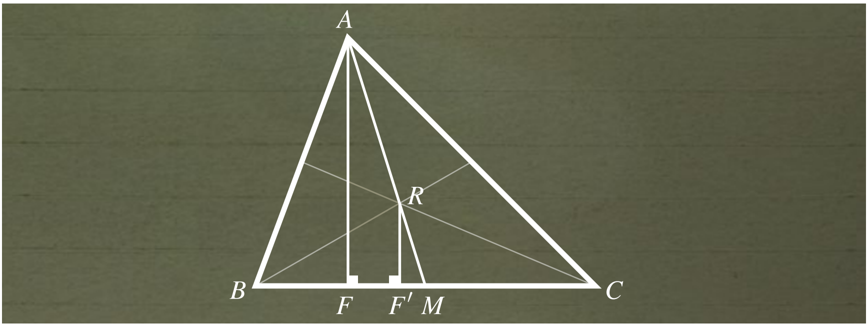
The trilinear coordinates of the centroid of $\triangle ABC$ are

$$[|AB| \cdot |AC| : |BA| \cdot |BC| : |CA| \cdot |CB|].$$

Proof. First the labels:

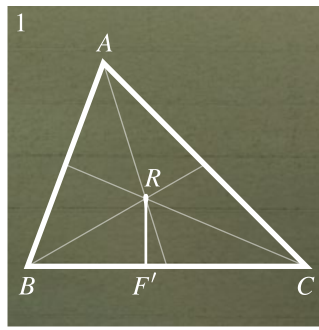
F: the foot of the altitude through A;
M: the midpoint of the side BC;
R: the centroid of $\triangle ABC$ (the intersection of the medians);
F': the foot of the perpendicular through R to the side BC.

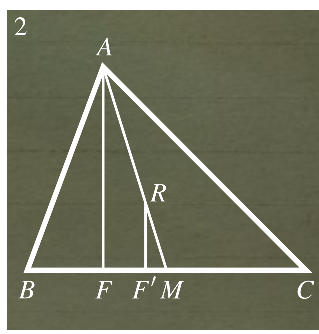
In addition, just for convenience write $a = |BC|$, $b = |AC|$, and $c = |AB|$.

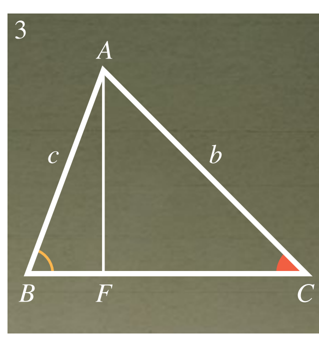


The last few results relied upon some essential property of the center in question– for the circumcenter it was the fact that it is equidistant from the three vertices; for the orthocenter, that it is the circumcenter of a larger triangle. This argument also draws upon such a property– that the centroid is located 2/3 of the way down a median from the vertex. Let's look at $[R,BC]$ which is one of the signed distances needed for the trilinear coordinates.

$$[R,BC] \underset{①}{=} |RF'| \underset{②}{=} \tfrac{1}{3}|AF| \underset{③}{=} \tfrac{1}{3}c\sin(\angle B) = \tfrac{1}{3}b\sin(\angle C)$$







1. Unlike the circumcenter and orthocenter, the median is always in the interior of the triangle, even when the triangle is right or obtuse. Therefore the signed distance $[R,BC]$ is the positive distance $|R,BC|$. Since RF' is the perpendicular to BC that passes through R, $|RF'|$ measures that distance.
2. This is the key step. Between the median AM and the parallel lines AF and RF' there are two triangles, $\triangle AFM$ and $\triangle RF'M$. These triangles are similar by A·A (they share the angle at M and the right angles at F and F' are congruent). Furthermore, because R is located 2/3 of the way down the median from the vertex, $|RM| = \frac{1}{3}|AM|$. The legs of those triangles must be in the same ratio, so $|RF'| = \frac{1}{3}|AF|$.
3. The goal is to relate $|AF|$ to the sides and angles of the original triangle, and we can now easily do that in two ways. In the right triangle $\triangle AFB$,

   $$\sin(\angle B) = \frac{|AF|}{c} \implies |AF| = c\sin(\angle B),$$

   and in the right triangle $\triangle AFC$,

   $$\sin(\angle C) = \frac{|AF|}{b} \implies |AF| = b\sin(\angle C).$$

Similarly, we can calculate the distances to the other two sides as

$$[R,AC] = \tfrac{1}{3}a\sin(\angle C) = \tfrac{1}{3}c\sin(\angle A)$$
$$[R,AB] = \tfrac{1}{3}b\sin(\angle A) = \tfrac{1}{3}a\sin(\angle B)$$

and so the exact trilinear coordinates of the centroid can be written as

$$R = \left[\tfrac{1}{3}c\sin(\angle B) : \tfrac{1}{3}a\sin(\angle C) : \tfrac{1}{3}b\sin(\angle A)\right].$$

There is still a little more work to get to the more symmetric form presented in the theorem. Note from the calculation in step (3) above, that,

$$c\sin(\angle B) = b\sin(\angle C) \implies \frac{\sin(\angle B)}{b} = \frac{\sin(\angle C)}{c}$$

Likewise, the ratio $\sin(\angle A)/a$ also has that same value (this is the "law of sines"). Therefore we can multiply by the value $3b/\sin(\angle B)$ in the first coordinate, $3c/\sin(\angle C)$ in the second coordinate, and $3a/\sin(\angle A)$ in the third coordinate, and since they are all equal, the result is an equivalent set of trilinear coordinates for the centroid $R = [bc : ca : ab]$. $\square$

To close out this lesson, and as well this section of the book, I want to make passing reference to another triangular coordinate system called barycentric coordinates. The trilinear coordinates that we have just studied put the incenter at the center of the triangle in the sense that it is the one point where are three coordinates are equal. With barycentric coordinates, that centermost point is the centroid. This is useful because if the triangle is a flat plate with a uniform density, then the centroid marks the location of the center of mass (the balance point). The barycentric coordinates of another point, then, give information about how to redistribute the mass of the plate so that *that point* is the balance point. Barycentric coordinates are usually presented in conjunction with the trilinear coordinates as the two are closely related. I am not going to do that though because I think we need to talk about area first, and area is still a ways away.

Exercises

1. (On the existence of similarly-positioned points) Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are similar, with scaling constant k, so that

$$|AB| = k|AB| \quad |B'C'| = k|BC| \quad |C'A'| = k|CA|.$$

Given any point P, show that there exists a unique point P so that

$$[A'P'] = k[AP] \quad [B'P'] = k[BP] \quad [C'P'] = k[CP].$$

2. (On the uniqueness of trilinear coordinate representations) For a given triangle $\triangle ABC$, is it possible for two distinct points P and Q to have the same trilinear coordinates?

3. What are the trilinear coordinates of the three excenters of a triangle?

4. Show that the trilinear coordinates of the center of the nine-point circle of $\triangle ABC$ are

$$[\cos((\angle B) - (\angle C)) : \cos((\angle C) - (\angle A)) : \cos((\angle A) - (\angle B))].$$

This one is a little tricky, so here is a hint if you are not sure where to start. Suppose that $\angle B$ is larger than $\angle C$. Label

O: the center of the nine-point circle,
P: the circumcenter,
M: the midpoint of BC, and
X: the foot of the perpendicular from O to BC.

The key is to show that the angle $\angle POX$ is congruent to $\angle B$ and that $\angle POM$ is congruent to $\angle C$. That will mean $(\angle MOX) = (\angle B) - (\angle C)$.